Optimal ratcheting of dividends in insurance

Hansjoerg Albrecher (University of Lausanne)

In this talk, we give an overview of recent developments in identifying optimal dividend payment strategies for an insurance company, when the dividend rate is not allowed to decrease. The optimality criterion here is to maximize the expected value of the aggregate discounted dividend payments up to the time of ruin. In the framework of the classical risk model and its Brownian approximation, the solution of the corresponding two-dimensional optimal control problem is presented and optimal strategies are numerically determined for several concrete examples.
The implementations illustrate that the restriction of ratcheting does not lead to a large efficiency loss when compared to the classical unconstrained optimal dividend strategy. We also consider an extension of the results to drawdown constraints on the dividend rate, where a curious square-root rule emerges.

ON EXPONENTIAL ALMOST SURE SYNCHRONIZATION OF A ONE-DIMENSIONAL DIFFUSION WITH NONREGULAR DRIFT

OLGA ARYASOVA

The main object of our study is a one-dimensional stochastic differential equation (SDE) of the type

$$\begin{cases} d\varphi_t = (-\lambda\varphi_t + a(\varphi_t))\,dt + \sigma(\varphi_t)\,dw_t,\ t > 0,\\ \varphi_0 \equiv x, \end{cases}$$

where λ is a positive real number, the drift a is measurable, the diffusion coefficient σ is a Lipschitz continuous non-degenerate function, and $(w_t)_{t\geq 0}$ is a Wiener process.

Thanks to the celebrated transform method, Zvonkin proved in [1] that this SDE admits a unique strong solution $(X_t^x)_{t>0}$. Moreover, it was proved during the last decade that due to the presence of noise, the flow $(X_t^x)_{t\geq 0, x\in\mathbb{R}}$ shows good spatial-regularity properties even if the drift function is discontinuous. Concerning the asymptotic stability of the flow there are much less results in the literature.

We solve the question of almost sure synchronization in high dissipative regime (λ large). We prove that two trajectories of that diffusion converge almost surely to one another at an exponential explicit rate as soon as the dissipative coefficient is large enough. The result is obtained for a wide class of SDEs with irregular drift functions: the function a is only supposed to be the sum of a Lipschitz function and of a bounded measurable one. Furthermore, we exhibit an explicit exponential convergence rate to 0 for $|X_t^x - X_t^y|$, both almost surely and in L_p. To our knowledge it is the first result of that type under such general assumptions.

In the spirit of Zvonkin, our approach is based on an accurately chosen space-transform in such a way that the transformed SDE - written via the new coordinate - has a simpler structure. A similar method could theoretically be used in more general context - multidimensional diffusions or Lévy-noise. However, the construction of corresponding transforms requires the investigation of elliptic equations whose solution is a non-trivial problem.

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Parameter estimation in stochastic heat equation with fractional Brownian motion

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We study stochastic heat equations with three types of noises: white noise, fractional Brownian noise and a mixed fractional Brownian noise. We investigate the covariance structure, stationarity, and asymptotic behavior of the solution for each case.

For the stochastic heat equation with white noise we construct a strongly consistent and asymptotically normal estimator of diffusion parameter.

For the equation driven by a fractional Brownian motion we construct strongly consistent estimators of two unknown parameters, namely, the diffusion parameter σ and the Hurst parameter $H \in (0,1)$. We also prove joint asymptotic normality of the estimators in the case $H \in (0, \frac{3}{4})$.

For the stochastic heat equation with mixed fractional Brownian motion we construct a strongly consistent estimator for the Hurst index H and prove its asymptotic normality for $H < 3/4$. Then assuming the parameter H to be known, we deal with joint estimation of the coefficients at Wiener process and at fractional Brownian motion.

PROPERTIES OF UTILITY MAXIMIZATION FUNCTIONALS FOR NON-CONCAVE UTILITY FUNCTION IN COMPLETE MARKET MODEL

OLENA BAHCHEDJIOGLOU AND GEORGIY SHEVCHENKO

This work is devoted to the study of the utility maximization problem. There are a lot of different aspects which can be considered while solving the optimization problem, such as completeness of the market, properties of the utility function, model settings, modeling of the payoff, and so on. We consider the complete market model, non-decreasing upper-semicontinuous non-concave utility function satisfying mild growth condition, and study the standard and constrained optimization problems while considering both the standard and robust utility maximization problems.

We proved the existence and uniqueness of the optimal solution to the standard non-concave utility maximization problem and constructed its explicit form under the assumption of standard budget constraints. It was shown that this solution is also a unique optimal solution for the maximization problem of the concavified utility function.

In the case of implementing an additional upper bound given by some random variable, we proved a similar theorem if the given random variable is discrete. Moreover, we presented examples that show that previous conclusions may fail in the case of a continuous random variable that represents an upper bound.

Subsequently, in the case of a robust utility maximization problem deriving the optimal solution is based on the study of the minimax identity for the initial non-concave utility function. We obtained equalities and inequalities to relate the robust utility functional of the initial utility function and its concavification and derived the assumptions under which minimax identity holds for the initial utility function. Besides, similar results were obtained in the case with an additional upper bound on the budget, represented, as before, by some random variable. The crucial step for obtaining the mentioned results with implementing an additional upper bound is the use of the regular conditional distribution which sheds new light on the possible approaches for solving the optimization problem.

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Large-scale Wasserstein gradient flows with applications for computing diffusions

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Wasserstein gradient flows provide a powerful means of understanding and solving many diffusion equations. Specifically, Fokker-Planck equations, which model the diffusion of probability measures, can be understood as gradient descent over entropy functionals in Wasserstein space. This equivalence, introduced by Jordan, Kinderlehrer and Otto, inspired the so-called JKO scheme to approximate these diffusion processes via an implicit discretization of the gradient flow in Wasserstein space. Solving the optimization problem associated to each JKO step, however, presents serious computational challenges. We introduce a scalable method to approximate Wasserstein gradient flows, targeted to machine learning applications. Our approach relies on input-convex neural networks (ICNNs) to discretize the JKO steps, which can be optimized by stochastic gradient descent. Unlike previous work, our method does not require domain discretization or particle simulation. As a result, we can sample from the measure at each time step of the diffusion and compute its probability density. We demonstrate our algorithm's performance by computing diffusions following the Fokker-Planck equation and apply it to unnormalized density sampling as well as nonlinear filtering.

Joint calibration of SPX and VIX options with signature-based models

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We consider a stochastic volatility model where the dynamics of the volatility are described by linear functions of the (time extended) signature of a primary underlying process, which is supposed to be some multidimensional continuous semimartingale. Under the additional assumption that this primary process is of polynomial type, we obtain closed form expressions for the VIX squared, exploiting the fact that the truncated signature of a polynomial process is again a polynomial process. Adding to such a primary process the Brownian motion driving the stock price, allows then to express both the log-price and the VIX squared as linear functions of the signature of the corresponding augmented process. This feature can then be efficiently used for pricing and calibration purposes. Indeed, as the signature samples can be easily precomputed offline, the calibration task can be split into offline sampling and a standard optimization. For both the SPX and VIX options we obtain highly accurate calibration results, showing that this model class allows to solve the joint calibration problem without adding jumps or rough volatility, but just path-dependence via the signature process.

Backward Stochastic Differential Equations with interaction

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Backward stochastic differential equations with interaction (shorter BSDEs with interaction) are introduced. Existence and uniqueness result for BSDE with interaction is proved under version of Lipschitz condition with respect to Wasserstein distance. Such kind of BSDE arises naturally when considering the Monge-Kantorovich problem. In the proof we start from discrete measures using known result of Pardoux and Peng and approximate general measure via Wasserstein distance.

Second Order Random Fields and Yield Curve Modeling

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The calibration of yield curve models imply the difficult task of estimating the covariance structure of the rates at the various maturities. This covariance structure drives the shape of risk factors in the HJM or BGM models. Expanding to infinite dimensional models, using either SPDEs or cylindrical Brownian motions, the shape of risk factors becomes even more unstable. Forcing a finite dimensional model, even more so, forcing a Markovian constraint on the dynamics leads to instabilities when recalibrating the model. We introduce a different type of constraint on the covariance structure, based on random fields after a change of variable in the range of maturities. This provides a very stable variance structure and provides a framework which is easy to expand to infinite dimensions.
We then propose to make this volatility structure stochastic in the space of Hilbert-Schmidt operators and state an existence result of mild solutions when a smoothing term is introduced in the yield curve dynamics.

Joint work with Zeyu Cao.

New Exact Solutions for PDEs with Mixed Boundary Conditions

Martino Grasselli

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We develop methods for the solution of inhomogeneous Robin type boundary value problems (BVPs) that arise for certain linear parabolic Partial Differential Equations (PDEs) on a half line, as well a second order generalisation. We are able to obtain non-standard solutions to equations arising in a range of areas, including mathematical finance, stochastic analysis, hyperbolic geometry and mathematical physics. Our approach uses the odd and even Hilbert transform methods. The solutions we obtain and the method itself seem to be new.

Joint work with Mark Craddock and Andrea Mazzoran

Optimal stopping: Bermudan strategies meet non-linear evaluations

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January 6, 2023

Abstract:

We address an optimal stopping problem over the set of Bermudan-type strategies Θ (which we understand in a more general sense than the stopping strategies for Bermudan options in finance) and with non-linear operators (non-linear evaluations) assessing the rewards, under general assumptions on the non-linear operators ρ. We provide a characterization of the value family V in terms of what we call the (Θ, ρ)-Snell envelope of the the pay-off family. We establish a Dynamic Programming Principle. We provide an optimality criterion in terms of a (Θ, ρ)-martingale property of V on a stochastic interval. We investigate the (Θ, ρ)-martingale structure and we show that the "first time" when the value family coincides with the pay-off family is optimal. The reasoning simplifies in the case where there is a finite number n of pre-described stopping times, where n does *not* depend on the scenario ω. We provide examples of non-linear operators entering our framework.

Paolo Guasoni

Title: Rogue Traders

Abstract:

Investing on behalf of a firm, a trader can feign personal skill by committing fraud that with high probability remains undetected and generates small gains, but that with low probability bankrupts the firm, offsetting ostensible gains. Honesty requires enough skin in the game: if two traders with isoelastic preferences operate in continuous-time and one of them is honest, the other is honest as long as the respective fraction of capital is above an endogenous fraud threshold that depends on the trader's preferences and skill. If both traders can cheat, they reach a Nash equilibrium in which the fraud threshold of each of them is lower than if the other one were honest. More skill, higher risk aversion, longer horizons, and greater volatility all lead to honesty on a wider range of capital allocations between the traders.

https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3870658

Ruin problems with investments on a finite interval: PIDEs and their viscosity solutions

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We study the ruin problem when an insurance company invests its reserve in a risky asset whose the price dynamics is given by a geometric Lévy process. We show that the ruin probabilities on a finite interval satisfy a partial integro-differential equation understood in the viscosity sense and prove a result on the uniqueness of solution for a boundary value problem.

Joint work with Viktor Antipov.

Mild to classical solutions for XVA equations under stochastic volatility

Damiano Brigo* Federico Graceffa** Alexander Kalinin §

December 23, 2021

Abstract

We extend the valuation of contingent claims in presence of default, collateral and funding to a random functional setting and characterise pre-default value processes by martingales. Pre-default value semimartingales can also be described by BSDEs with random path-dependent coefficients and martingales as drivers. En route, we generalise previous settings by relaxing conditions on the available market information, allowing for an arbitrary default-free filtration and constructing a broad class of default times. Moreover, under stochastic volatility, we characterise pre-default value processes via mild solutions to parabolic semilinear PDEs and give sufficient conditions for mild solutions to exist uniquely and to be classical.

MSC2010 classification: 91G20, 91G80, 60G40, 60H20, 60H30, 35K58.
Keywords: XVA, valuation, collateral, funding costs, default time, stochastic volatility, stochastic differential equation, mild solution, semilinear parabolic PDE.

1 Introduction

The aim of this paper is to address the valuation of contingent claims in a financial market under default risk, collateralisation and funding costs and benefits. Based on a general probabilistic setting, we develop a market model from previous works that consists of an investor and a counterparty entering a derivative contract. To evaluate such an agreement with default-free information only, we derive a nonlinear pre-default valuation equation and characterise its solutions, the pre-default value processes.

By focusing on a stochastic volatility model for the underlying risky asset and its generalised variance, or simply quasi variance, we will reach a parabolic semilinear partial differential equation (PDE) that establishes a direct relation between pre-default value processes and mild solutions. While pursuing this goal, we will achieve further extensions of preceding papers in this area, and the two articles [11] and [12] in particular, that focused on viscosity and classical solutions. Our main contributions to the existing literature can be described as follows:

(1) *The available market information may fail to provide any knowledge about the first time of default.* In the earlier work [12] even full insight into the separate default times of the investor and the counterparty was required, as the two latter filtrations in (3.1) were supposed to be equal.

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(2) To handle the general relation (3.1) between the two filtrations that model the default-free and the available market information, *a variety of representations for conditional expectations* is derived in Section 2.2. Thereby, Corollary 2.2 explains how to identify random quantities before the first time of default occurs and *the default-free filtration is arbitrary.* In particular, it does not need to coincide with the (augmented) natural filtration of a diffusion.

(3) *The default times of the two parties, except being conditionally independent and admitting a distribution satisfying weak regularity conditions, are arbitrary.* This is based on an explicit construction in Section 2.3, which allows for *a detailed analysis of default times*, including a formula for their survival functions in Proposition 2.10. Hitting times that involve a gamma distribution, or more specifically, an exponential distribution, as considered for example in [12], are feasible, as shown in Example 2.12, and we refer to [21] for a discussion on related issues on immersion.

(4) *The pre-default valuation equation* (VE) *that only requires default-free information is deduced from the generalised valuation equation* (3.14) in Proposition 3.5. To this end, for all cash flows, costs and benefits appearing in the valuation, we give concise financial interpretations and state the necessary measurability, path regularity and integrability conditions in Sections 3.2 and 3.3.

(5) We give *two characterisations for pre-default value processes*, the solutions to (VE). While Proposition 3.6 relates pre-default valuation with the martingale property of the process in (3.20), Corollary 3.8 describes value processes that are semimartingales by the BSDE (3.22) with random path-dependent coefficients, driven by a martingale and analysed in Proposition 3.7. In the previous work [15], for instance, necessary and sufficient conditions for the existence of solutions to the pre-default valuation equation were not explicitly given.

(6) *A stochastic volatility model*, described by the two-dimensional SDE (4.1), is introduced in Section 4.1. Regarding the quasi variance process, we give *a criterion for solutions to one-dimensional SDEs to have a.s. positive paths* in Proposition 4.4, by extending the main result in [29], and demonstrate in Example 4.5 that sums of power functions such as (4.10) may appear as drift and diffusion coefficients. Combined with the uniqueness and existence results from [26], Proposition 4.6 proves that the transformed SDE (4.11), obtained by taking the log-price process, is uniquely solvable and yields a diffusion. Then, Example 4.7 applies this result to the specific SDE (4.13), which extends the Heston model [23] and the Garch diffusion model [28].

(7) By imposing the dynamics (4.1) on the price process and its quasi variance in the market model from Section 3, we eventually reach the parabolic semilinear PDE (4.20). One of the main achievements of the article is that *we characterise pre-default value processes by means of mild solutions* to (4.20) in Theorem 4.10. As the derived diffusion serves as Makov process in the setting of [25], we obtain unique bounded mild solutions in Proposition 4.12 and, under the conditions of Corollary 4.14, mild solutions are in fact classical.

To the best of our knowledge, our paper is the first work on nonlinear valuation and XVA equations to propose mild solutions as middle ground between viscosity and classical solutions for parabolic semilinear valuation PDEs, including also stochastic volatility.

While viscosity solutions can be described by means of test functions to bypass a priori considerations regarding differentiability, mild solutions stem from related implicit integral equations and allow for Picard iterations, as Proposition 2.12 in [25] shows, for example. Further, the mild solution concept leads to general derivative formulas, deduced in [32], using Lemma 1.1, Corollary 2.8 and Theorems 2.9 and 3.2 therein. In this sense mild solutions are more tractable than those of viscosity type. If a parabolic semilinear PDE, which may also be path-dependent, admits continuous coefficients, then, under certain linear and polynomial growth conditions and a Lipschitz condition, Corollary 4.17 in [17] asserts that the two notions coincide. The valuation PDE that we derive, however, does not meet these regularity conditions and, according to the characterisation that we find in Theorem 4.10, the mild solution concept is indeed suitable.

Note that the mathematical and modelling achievements of this paper are not obtained in the most general setting in terms of financial adjustments, as we do not include capital valuation adjustments and initial margins in our analysis. However, we believe that the default, collateral as variation margin and funding effects we are considering are more than sufficient to highlight the mathematical difficulties of these nonlinear valuation problems. For this purpose, we would like to contextualize this work in the broad area of nonlinear valuation and valuation adjustments, or 'XVA'.

Prior to the financial crisis of 2007-2008, financial institutions at times ignored the credit risk of highly-rated counterparties in valuing and hedging contingent claims. Then, in a short period of about one month, around October 2008, eight mainstream financial institutions defaulted (Fannie Mae, Freddie Mac, Lehman Brothers, Washington Mutual, Landsbanki, Glitnir and Kaupthing, to which we could also add Merrill Lynch that was saved through a merge with Bank of America). This highlighted dramatically the fact that no institution could be considered default-free, no matter how systemic or prestigious. This forced dealers and financial institutions to reassess the valuation of contingent claims, leading to a much more widespread adoption of collateralisation, through various adjustments to their book value.

We will now list some of these adjustments as separate effects, but one should keep in mind that the nonlinearity of the valuation equations makes this separation quite artificial. In any case it is difficult to do justice to the entire literature on such valuation adjustments. For a full introduction to credit and funding valuation adjustments and all related references we refer to the first chapter of either [14] or [18]. Here we will only provide a quick summary for context and a few references, before moving to the full nonlinear valuation equation and its analysis.

Firstly, the credit valuation adjustment (CVA) has been introduced to correct the value of a trade with the expected costs borne by one dealer in scenarios where its counterparty defaults. CVA had been around for some time, see for example [13], and its most sophisticated version can include credit migration and ratings transition, as shown in [5]. Further, it already leads to BSDEs under replacement closeout, which was taken into account in [20] and [15]. It is worth pointing out that collateralisation has not completely eliminated CVA. In [10], for instance, it is shown that for some particular deals gap risk may leave a quite large CVA even in presence of daily collateralisation. This is one of the reasons for the introduction of the initial margin as a further collateralisation tool supplementing the variation margin.

Secondly, the debit valuations adjustment (DVA), that on one hand is simply CVA seen from the other side, corrects the price with the expected benefits to the dealer due to scenarios where the dealer has an early default on the trade. DVA may lead to a controversial profit that can be booked when the credit quality of the dealer deteriorates, which has led a discussion on considering it more of a funding benefit than a debit adjustment. While the Basel Committee has made recommendations against the use of DVA, accounting standards by the FASB accept DVA for fair value. A detailed discussion can be found in [14]. On top of this, DVA is very difficult to hedge, as this would involve selling protection on oneself, and this is a further reason why regulators opposed it.

After CVA and DVA, the funding valuation adjustment (FVA) was introduced. FVA is the price adjustment due to the cost of funding the trading activity surrounding a trade. To maintain a trade, the trading desk needs to borrow funds from the bank treasury, giving back funds occasionally. All borrowing and lending has a cost or remuneration in terms of interest fees, and this has to be accounted for. Following FVA, a capital valuation adjustment (KVA) has started being discussed for the cost of capital one has to set aside in order to be able to trade. We will not address KVA here, since its very definition is currently subject to intense debate in the industry. Instead, we refer to [19] for a recent work addressing the cost of capital. A further adjustment that has been considered is a charge for the cost of setting up the initial margin for a trade. This is often called margin valuation adjustment, or MVA, and was assessed for example in [15] and more recently in [3], where multiple curve effects are also discussed.

All such adjustments may concern both over the counter (OTC) derivatives trades and derivatives trades done through central clearing houses (CCP). These two cases are compared in [15], where the full mathematical structure of the problem of valuation under possibly asymmetric initial and variation margins, funding costs, liquidation delay and credit gap risk is explored. This nonlinear valuation analysis has been made more rigorous in the subsequent paper [12] and by many other authors.

For an early example of how asymmetric interest rates, even in absence of credit risk, lead to BSDEs see [22]. The paper [4] deals with the mathematical analysis of valuation equations in presence of all the above-mentioned effects and risks, except KVA. CVA and FVA are analysed in [7] in the area of life insurance contracts, and longevity swaps in particular. Finally, an in-depth discussion of replication in presence of default and funding effects is presented in [9], discussing also valuation in general settings when replication is not assumed. This article contributes to the literature on nonlinear valuation equations, from BSDEs to PDEs, of the type seen in the above-mentioned works, especially [20], [15], [4] and [12], by focusing on mild solutions among the other contributions listed earlier.

The paper is structured as follows. Section 2 sets up the notation and discusses the required probabilistic methods to handle the financial market model. Namely, after a concise introduction of the notation in Section 2.1, we deal with conditional expectations in Section 2.2 and provide a class of default times in Section 2.3.

In Section 3 we specify and analyse the market model. While Section 3.1 explains the setting, all the cash flows, costs and benefits that are relevant to determine the price of the derivative contract are quantified in Section 3.2. Then, in Section 3.3 the pre-default valuation equation (VE) is derived and its solutions, the pre-default value processes, are characterised in Proposition 3.6 and Corollary 3.8.

In Section 4 we impose a general stochastic volatility model on the underlying risky asset and its quasi variance to deduce the pre-default valuation PDE (4.20). To this end, Section 4.1 considers the SDE (4.1) that governs the volatility model with regard to pathwise uniqueness, strong existence, moment estimates and positivity of paths. As a result, Proposition 4.6 shows that the transformed SDE (4.11) yields a diffusion.

Section 4.2 discusses a deterministic setting of the market model for the valuation PDE to prevail, and pre-default value processes are characterised by means of mild solutions in Theorem 4.10 there. An existence and uniqueness result for bounded mild solutions is derived in Proposition 4.12 and sufficient conditions for mild solutions to be classical are given in Corollary 4.14.

All proofs for the probabilistic methods in Section 2.2, the constructed hitting times in Section 2.3 and the market model of Section 3 are deferred to Section 5. The results for the volatility model and the valuation PDE in Section 4 are proven in Section 6.

2 Preliminaries

Throughout the paper, let $(\Omega, \mathscr{F}, P)$ denote a probability space, $T > 0$ and $(\mathscr{F}_t)_{t\in[0,T]}$, $(\tilde{\mathscr{F}}_t)_{t\in[0,T]}$ be two filtrations of $\mathscr{F}$.

2.1 Notation and basic concepts

We recall that the extended non-negative real line $[0,\infty]$ is completely metrizable in such a way that the resulting trace topology of $\mathbb{R}_+$ agrees with the topology on $\mathbb{R}_+$ induced by the absolute value function. For instance, take the metric given by

$$d_\infty(x,y) = |f_\infty(x) - f_\infty(y)|$$

for any $x, y \in [0,\infty]$ with the strictly increasing homeomorphism $f_\infty : \mathbb{R}_+ \to [0,1[$ given by $f_\infty(x) := x/(1+x)$ that satisfies $f_\infty(\infty) = 1$, where we set $f(\infty) := \lim_{x\uparrow\infty} f(x)$ for any real-valued monotone function f defined on some interval. We shall use the induced topology of d_∞ in Sections 2.2 and 2.3.

For $p \in [1,\infty[$ let $\mathscr{L}^p(\mathbb{R})$ denote the linear space of all real-valued Borel measurable p-fold Lebesgue integrable functions on $[0,T]$ and $\mathscr{L}^p(\mathbb{R}_+)$ stand for the convex cone of all $\mathbb{R}_+$-valued functions in $\mathscr{L}^p(\mathbb{R})$. For the Banach space of all real-valued càdlàg functions on $[0,T]$, endowed with the supremum norm, we use the standard notation $D([0,T])$.

Let $\mathscr{S}$ and $\tilde{\mathscr{S}}$ be the linear spaces of all (real-valued) processes that are adapted to $(\mathscr{F}_t)_{t\in[0,T]}$ and $(\tilde{\mathscr{F}}_t)_{t\in[0,T]}$, respectively, which will be used extensively in Section 3. Further, a real-valued function u on $[0,T]\times\mathbb{R}\times]0,\infty[$ will be called right-continuous if for each $(s,x,v) \in [0,T]\times\mathbb{R}\times]0,\infty[$ and any $\varepsilon > 0$ there is $\delta > 0$ such that

$$|u(s,x,v) - u(t,y,w)| < \varepsilon$$

for all $(t,y,w) \in [s,T]\times\mathbb{R}\times]0,\infty[$ with $|s-t| + |x-y| + |v-w| < \delta$. This notion of right-continuity in time and continuity in space from Definition 2.1 in [25] will be used for the right-hand Feller property of a diffusion in Section 4.

2.2 Representations of conditional expectations

In this section let $\mathscr{T}$ be a non-empty finite set of $[0,T]\cup\{\infty\}$-valued random variables. Each $\tau \in \mathscr{T}$ defines the smallest filtration $(\mathscr{H}_t^\tau)_{t\in[0,T]}$ under which it becomes a stopping time. Namely,

$$\mathscr{H}_t^\tau = \sigma\big(\mathbb{1}_{\{\tau\leq s\}} : s \in [0,t]\big) \quad \text{for all } t \in [0,T]. \tag{2.1}$$

By setting $\mathscr{H}_t := \bigvee_{\tau\in\mathscr{T}} \mathscr{H}_t^\tau$ for any $t \in [0,T]$, we obtain the smallest filtration under which any $\tau \in \mathscr{T}$ is a stopping time. Then the $(\mathscr{H}_t)_{t\in[0,T]}$-stopping time $\rho := \min_{\tau\in\mathscr{T}} \tau$ gives rise to the filtration $(\mathscr{F}_t^{\mathscr{T}})_{t\in[0,T]}$ defined via

$$\mathscr{F}_t^{\mathscr{T}} := \{\tilde{A} \in \mathscr{F} \mid \exists A \in \mathscr{F}_t : \{\rho > t\} \cap A = \{\rho > t\} \cap \tilde{A}\},$$

which satisfies $\mathscr{F}_t \vee \mathscr{H}_t \subset \mathscr{F}_t^{\mathscr{T}}$ for any $t \in [0,T]$. These concepts generalise the framework in [6][Section 5.1.1] and yield an essential relation between conditional expectations, given another filtration $(\tilde{\mathscr{F}}_t)_{t\in[0,T]}$ such that $\mathscr{F}_t \subset \tilde{\mathscr{F}}_t \subset \mathscr{F}_t \vee \mathscr{H}_t$ for all $t \in [0,T]$.

Lemma 2.1. *Any $[0,\infty]$-valued random variable X satisfies*

$$E[X\mathbb{1}_{\{\rho>t\}}|\tilde{\mathscr{F}}_s]P(\rho > s|\mathscr{F}_s) = E[X\mathbb{1}_{\{\rho>t\}}|\mathscr{F}_s]P(\rho > s|\tilde{\mathscr{F}}_s) \quad a.s.$$

for all $s, t \in [0,T]$ with $s \leq t$.

We notice that any decreasing sequence $(A_t)_{t\in[0,T]}$ in $\mathscr{F}$ satisfies $P(A_s|\mathscr{F}_s) \geq P(A_t|\mathscr{F}_s) = E[P(A_t|\mathscr{F}_t)|\mathscr{F}_s]$ a.s. for all $s,t \in [0,T]$ with $s \leq t$. In particular, for every random variable τ with values in $[0,T]\cup\{\infty\}$ we have

$$P(\tau > t|\mathscr{F}_t) = G_t(\tau) \quad \text{a.s.} \quad \text{for any } t \in [0,T] \tag{2.2}$$

and some $[0,1]$-valued $(\mathscr{F}_t)_{t\in[0,T]}$-supermartingale $G(\tau)$, which is called an *survival process* of τ relative to this filtration, unique up to a modification. This fact allows us to identify random variables before ρ occurs.

Corollary 2.2. *For $t \in [0,T]$ let X and $\tilde{X}$ be two $\mathbb{R}_+$-valued random variables that are measurable relative to $\mathscr{F}_t$ and $\tilde{\mathscr{F}}_t$, respectively. Then $X = \tilde{X}$ a.s. on $\{\rho > t\}$ if and only if*

$$XG_t(\rho) = E[\tilde{X}\mathbb{1}_{\{\rho>t\}}|\mathscr{F}_t] \quad a.s. \tag{2.3}$$

In this case, X is a.s. uniquely determined as soon as $G_t(\rho) > 0$ a.s.

Now we rewrite a conditional expectation of a stopped integral by means of the survival process $G(\rho)$.

Lemma 2.3. *Let $s \in [0,T]$ and $G(\rho)$ be measurable. If X and $\tilde{X}$ are two $[0,\infty]$-valued measurable processes such that X_t is $\mathscr{F}_t$-measurable and $X_t = \tilde{X}_t$ a.s. on $\{\rho > t\}$ for all $t \in [s,T]$, then*

$$E\bigg[\int_s^{T\wedge\rho} \tilde{X}_t\,dt\,\bigg|\,\mathscr{F}_s\bigg] = E\bigg[\int_s^T X_t G_t(\rho)\,dt\,\bigg|\,\mathscr{F}_s\bigg] \quad a.s.$$

To consider conditional expectations of processes combined with stopping times, we require a general concept of conditional independence.

Definition 2.4. Let $m \in \mathbb{N}$ and $\tau_1,\ldots,\tau_m$ be $[0,T]\cup\{\infty\}$-valued random variables. Then $\tau_1,\ldots,\tau_m$ are called $(\mathscr{F}_t)_{t\in[0,T]}$-*conditionally independent* if

$$P(\tau_1 > s_1,\ldots,\tau_m > s_m|\mathscr{F}_t) = P(\tau_1 > s_1|\mathscr{F}_t)\cdots P(\tau_m > s_m|\mathscr{F}_t) \quad \text{a.s.}$$

for each $t \in [0,T]$ and any $s_1,\ldots,s_m \in [0,t]$.

As $[0,T]\cup\{\infty\}$ is closed in the Polish space $[0,\infty]$, any random variable τ taking all its values there admits a regular conditional probability K given $\mathscr{F}_t$, where $t \in [0,T]$. That is, K is a Markovian kernel from $(\Omega,\mathscr{F}_t)$ to $[0,T]\cup\{\infty\}$ such that

$$P(\tau \in B|\mathscr{F}_t) = K(\cdot,B) \quad \text{a.s.} \quad \text{for any } B \in \mathscr{B}([0,T]\cup\{\infty\}).$$

In consequence, if two $[0,T]\cup\{\infty\}$-valued random variables are $(\mathscr{F}_t)_{t\in[0,T]}$-conditionally independent, then their joint conditional distribution with respect to $\mathscr{F}_t$ is completely determined up to time t in the following sense.

Lemma 2.5. *Let σ,τ be two $[0,T]\cup\{\infty\}$-valued $(\mathscr{F}_t)_{t\in[0,T]}$-conditionally independent random variables and $t \in [0,T]$. Then*

$$P((\sigma,\tau) \in C|\mathscr{F}_t)(\omega) = K(\omega,\cdot)\otimes L(\omega,\cdot)(C) \quad \textit{for } P\textit{-a.e. } \omega \in \Omega, \tag{2.4}$$

all $C \in \mathscr{B}(([0,t]\cup\{\infty\})^2)$ and any two respective regular conditional probabilities K and L of σ and τ given $\mathscr{F}_t$.

We conclude with the following integral representation within conditional expectations, which extends Proposition 5.11 in [6].

Proposition 2.6. *Let $s \in [0,T[$ and $\sigma, \tau \in \mathscr{T}$. Assume that $\tilde{X} \in \tilde{\mathscr{S}}$ admits bounded left-continuous paths such that the following three conditions hold:*

(i) $G(\sigma)$ is right-continuous and of finite variation and σ, τ are $(\mathscr{F}_t)_{t\in[0,T]}$-conditionally independent.

(ii) There exists an $(\mathscr{F}_t)_{t\in[0,T]}$-progressively measurable process X with bounded paths satisfying $X_t = \tilde{X}_t$ a.s. on $\{t < \sigma \leq T\}$ for each $t \in]s,T]$.

(iii) The paths $G(\tau)(\omega)$ and $X(\omega)$ are left-continuous except at countably many points, excluding any discontinuity point of $G(\sigma)(\omega)$, for each $\omega \in \Omega$.

If $\sup_{t\in]s,T]} |\tilde{X}_t| \mathbb{1}_{\{s<\sigma\leq T\wedge\tau\}}$ and $\sup_{t\in]s,T]} |X_t| G_t(\tau)(V_T(\sigma) - V_s(\sigma))$ are integrable, where $V(\sigma)$ is the variation process of $G(\sigma)$, then

$$E[\tilde{X}_T^\sigma \mathbb{1}_{\{s<\sigma\leq T\wedge\tau\}} | \mathscr{F}_s] = -E\bigg[\int_{]s,T]} X_t G_t(\tau)\, dG_t(\sigma) \bigg| \mathscr{F}_s\bigg] \quad a.s.$$

2.3 Construction of conditionally independent hitting times

For given $m \in \mathbb{N}$ let X be an $[0,\infty]^m$-valued $(\mathscr{F}_t)_{t\in[0,T]}$-adapted right-continuous process and ξ be an $\mathbb{R}_+^m$-valued $\tilde{\mathscr{F}}_0$-measurable random vector that is independent of $\mathscr{F}_T$ such that $\xi_1, \dots, \xi_m$ are independent.

We assume that the i-th coordinate process of X, denoted by $X^{(i)}$, is increasing, let G_i be the survival function of ξ_i and define a function τ_i on Ω with values in $[0,T] \cup \{\infty\}$ via

$$\tau_i := \inf\{t \in [0,T] \,|\, X_t^{(i)} \geq \xi_i\}$$

for any $i \in \{1,\dots,m\}$. Then the hitting time τ_i does not need to be an $(\mathscr{F}_t)_{t\in[0,T]}$-stopping time, as ξ_i may fail to be $\mathscr{F}_0$-measurable. However, the following facts hold.

Lemma 2.7. *The functions $\tau_1, \dots, \tau_m$ are $(\tilde{\mathscr{F}}_t)_{t\in[0,T]}$-stopping times that are conditionally independent relative to $(\mathscr{F}_t)_{t\in[0,T]}$ such that $\{\tau_j > t\} = \{X_t^{(j)} < \xi_j\}$ and*

$$P(\tau_1 > s_1, \dots, \tau_j > s_j | \mathscr{F}_t) = G_1(X_{s_1}^{(1)}) \cdots G_j(X_{s_j}^{(j)}) \quad a.s.$$

for any $j \in \{1,\dots,m\}$, each $t \in [0,T]$ and every $s_1, \dots, s_j \in [0,t]$.

As a direct consequence, $\rho := \min_{i\in\{1,\dots,m\}} \tau_i$ is an $(\tilde{\mathscr{F}}_t)_{t\in[0,T]}$-stopping time and each $(\mathscr{F}_t)_{t\in[0,T]}$-survival process $G(\rho)$ of ρ satisfies

$$G_s(\rho) = P(\rho > s | \mathscr{F}_t) = G_1(X_s^{(1)}) \cdots G_m(X_s^{(m)}) \quad \text{a.s.} \tag{2.5}$$

for any $s, t \in [0,T]$ with $s \leq t$. Further relevant properties may be inferred by using $a \in \mathbb{R}_+^m$ and $b \in [0,\infty]^m$ defined coordinatewise via $a_i := \operatorname{ess\,inf} \xi_i$ and $b_i := \operatorname{ess\,sup} \xi_i$.

Lemma 2.8. *For each $s \in [0,T]$ the following three assertions hold:*

(i) $G_s(\rho) > 0$ a.s. $\Leftrightarrow$ $X_s^{(i)} < b_i$ a.s. for all $i \in \{1,\dots,m\}$.

(ii) $\rho \leq s$ a.s. $\Leftrightarrow$ $X_s^{(i)} \geq b_i$ for some $i \in \{1,\dots,m\}$ a.s. Similarly,

$$\rho > s \quad a.s. \quad \Leftrightarrow \quad X_s^{(i)} \leq a_i \quad a.s., \quad if \quad \xi_i > a_i \quad a.s.,$$

and $X_s^{(i)} < a_i$ a.s., if $P(\xi_i = a_i) > 0$, for any $i \in \{1,\dots,m\}$.

(iii) $\rho \neq s$ a.s. whenever $s > 0$, X is a.s. continuous and $G_1, \dots, G_m$ are continuous.

Example 2.9. Let $\hat{x} \in \mathbb{R}_+^m$ and λ be an $[0,\infty]^m$-valued process that is progressively measurable relative to $(\mathscr{F}_t)_{t\in[0,T]}$ such that

$$X_t = \hat{x} + \int_0^t \lambda_s \, ds \quad \text{for all } t \in [0,T]. \tag{2.6}$$

Then X is left-continuous, by monotone convergence, and the assumed right-continuity of X holds if and only if for every $\omega \in \Omega$ there is $t_\omega \in]0,T]$ such that

$$\sum_{i=1}^m \int_0^{t_\omega} \lambda_s^{(i)}(\omega) \, ds < \infty.$$

In addition, Lemma 2.8 entails the following three statements:

(1) Assume that $b_i = \infty$ for all $i \in \{1,\dots,m\}$. Then $G_s(\rho) > 0$ a.s. for any $s \in [0,T]$ if and only if λ admits a.s. (Lebesgue) integrable paths, and

$$\rho < \infty \quad \text{a.s.} \quad \Leftrightarrow \quad \sum_{i=1}^m \int_0^T \lambda_t^{(i)} \, dt = \infty \quad \text{a.s.}$$

(2) If $\hat{x}_i \geq a_i$ for any $i \in \{1,\dots,m\}$ and the event of all $\omega \in \Omega$ with $\int_0^{t_\omega} \lambda_s^{(i)}(\omega)\, ds > 0$ for all $i \in \{1,\dots,m\}$ has positive probability, then $P(\rho > s) < 1$ for any $s \in]0,T]$.

(3) $\rho \neq 0$ a.s. $\Leftrightarrow$ For each $i \in \{1,\dots,m\}$ we have $\hat{x}_i \leq a_i$ with equality if and only if $\xi_i > a_i$ a.s. Further, $\rho \neq s$ a.s. for all $s \in]0,T]$ if $G_1,\dots,G_m$ are continuous.

Note that if $X_0^{(i)} \geq a_i$ a.s. for some $i \in \{1,\dots,m\}$ and ξ_i is a.s. constant, which is equivalent to the condition that $a_i = b_i$, then $\rho = 0$ a.s., since $P(\rho > 0) \leq G_i(b_i) = 0$. For this reason, let us now assume that $a_i < b_i$ for all $i \in \{1,\dots,m\}$.

We define an event in $\mathscr{F}_t$ by $\Lambda_t := \bigcap_{i=1}^m \{X_t^{(i)} < b_i\}$ for each $t \in [0,T]$. While $\{\rho > t\}$ is included in Λ_t, we have $P(\rho > t) > 0$ if and only if $P(\Lambda_t) > 0$, by Lemmas 2.7 and 2.8. Based on these considerations, we provide a formula for the survival function of ρ.

Proposition 2.10. *Let $\hat{x} \in \mathbb{R}_+^m$ and λ be some $[0,\infty]^m$-valued $(\mathscr{F}_t)_{t\in[0,T]}$-progressively measurable process satisfying* (2.6) *such that for each $\omega \in \Omega$ we have*

$$0 < \int_0^{t_\omega} \lambda_s^{(i)}(\omega)\, ds < \infty \quad \text{for all } i \in \{1,\dots,m\} \text{ and some } t_\omega \in]0,T]. \tag{2.7}$$

If $\hat{x}_i \in [a_i, b_i[$ and G_i is continuously differentiable on $]a_i,b_i[$ for any $i \in \{1,\dots,m\}$, then

$$\begin{aligned} P(\rho > t) &= G_1(\hat{x}_1)\cdots G_m(\hat{x}_m) P(\Lambda_t) \\ &\quad + \int_0^t E\Big[G_1(X_s^{(1)})\cdots G_m(X_s^{(m)}) \sum_{i=1}^m \lambda_s^{(i)} \Big(\frac{G_i'}{G_i}\Big)(X_s^{(i)}); \Lambda_t\Big]\, ds \end{aligned}$$

for every $t \in [0,T]$.

Remark 2.11. The pathwise stated condition (2.7) is satisfied if and only if for any $\omega \in \Omega$ there are $t_\omega \in]0,T]$ and $s_\omega \in]0,t_\omega]$ such that

$$\lambda(\omega) > 0 \quad \text{a.e. on }]0,s_\omega[\quad \text{and} \quad \lambda(\omega) \text{ is integrable on } [0,t_\omega].$$

For instance, this is the case is if λ is $]0,\infty[^m$-valued and admits integrable paths.

To conclude our analysis, let us impose the gamma distribution on $\xi_1,\dots,\xi_m$. This includes the hitting times considered in [12] as special case, by choosing an exponential distribution with mean one.

Example 2.12. For each $i \in \{1, \dots, m\}$ let ξ_i be gamma distributed with shape $\alpha_i > 0$ and rate $\beta_i > 0$. That is, its survival function and the gamma function Γ satisfy

$$G_i(x) = \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} \int_x^\infty y^{\alpha_i - 1} e^{-\beta_i y}\, dy \quad \text{for all } x \in \mathbb{R}_+.$$

We suppose that λ is an $[0,\infty]^m$-valued $(\mathscr{F}_t)_{t\in[0,T]}$-progressively measurable process such that $X_t = \int_0^t \lambda_s\, ds$ for all $t \in [0,T]$ and (2.7) holds. Then the formula (2.5) yields that

$$P(\rho > s|\mathscr{F}_t) = \frac{\gamma(\alpha_1, \beta_1 X_s^{(1)})}{\Gamma(\alpha_1)} \cdots \frac{\gamma(\alpha_m, \beta_m X_s^{(m)})}{\Gamma(\alpha_m)} \quad \text{a.s.}$$

for any $s, t \in [0,T]$ with $s \leq t$, where $\gamma :]0,\infty[^2 \to]0,\infty[$, $\gamma(\alpha, x) := \int_x^\infty y^{\alpha-1} e^{-y}\, dy$ is the upper incomplete gamma function. From Example 2.9 we in particular infer that

$$P(\rho > s) < 1 \quad \text{for all } s \in]0,T] \quad \text{and} \quad \rho \neq s \quad \text{a.s.} \quad \text{for any } s \in [0,T].$$

Moreover, if λ admits integrable paths, then $P(\rho = \infty) > 0$ and Proposition 2.10 entails that the distribution of ρ decomposes into its continuous and discrete part. Namely,

$$P(\rho \in B) = \int_{B\cap[0,T]} \varphi_\rho(s)\, ds + \bigg(1 - \int_0^T \varphi_\rho(s)\, ds\bigg)\delta_\infty(B)$$

for any $B \in \mathscr{B}([0,T] \cup \{\infty\})$ with the measurable integrable function $\varphi_\rho : [0,T] \to [0,\infty]$ given by

$$\varphi_\rho(s) := E\bigg[\frac{\gamma(\alpha_1, \beta_1 X_s^{(1)})}{\Gamma(\alpha_1)} \cdots \frac{\gamma(\alpha_m, \beta_m X_s^{(m)})}{\Gamma(\alpha_m)} \sum_{i=1}^m \lambda_s^{(i)} \frac{\beta_i^{\alpha_i} (X_s^{(i)})^{\alpha_i - 1}}{\gamma(\alpha_i, \beta_i X_s^{(i)})} e^{-\beta_i X_s^{(i)}}\bigg].$$

3 A financial market model with default

We aim to evaluate a derivative contract between an *investor* $\mathcal{I}$ and a *counterparty* $\mathcal{C}$, both considered as financial entities, with a special focus on the case that $\mathcal{I}$ stands for an *investment bank* $\mathcal{B}$.

3.1 Model specifications

In the sequel, we interpret the two filtrations $(\mathscr{F}_t)_{t\in[0,T]}$ and $(\tilde{\mathscr{F}}_t)_{t\in[0,T]}$ as the *temporal developments of the default-free information* and the *whole available information* on an underlying financial market, respectively.

We use two $[0,T] \cup \{\infty\}$-valued random variables $\tau_\mathcal{I}$ and $\tau_\mathcal{C}$ to model the respective *default times* of the investor $\mathcal{I}$ and the counterparty $\mathcal{C}$. Then $\tau := \tau_\mathcal{I} \wedge \tau_\mathcal{C}$ stands for the *time of a party to default first*. By using the notation in (2.1), we require that

$$\mathscr{F}_t \subset \tilde{\mathscr{F}}_t \subset \mathscr{F}_t \vee \mathscr{H}_t^{\tau_\mathcal{I}} \vee \mathscr{H}_t^{\tau_\mathcal{C}} \quad \text{for all } t \in [0,T]. \tag{3.1}$$

Thus, the available market information could yield no knowledge about the first time of default and it may fail to give any insight into the respective default times of $\mathcal{I}$ and $\mathcal{C}$.

In our continuous-time setting we assume that the distributions of $\tau_\mathcal{I}$ and $\tau_\mathcal{C}$ admit at most one atom, which is at infinity, and both parties cannot default simultaneously. That is, for any $t \in [0,T]$ we have

$$P(\tau_\mathcal{I} = t) = P(\tau_\mathcal{C} = t) = 0 \quad \text{and} \quad P(\tau_\mathcal{I} = \tau_\mathcal{C}, \tau < \infty) = 0. \tag{3.2}$$

The first condition implies that $\tau \neq t$ a.s. for all $t \in [0,T]$. However, as $\{\tau_\mathcal{I} = \tau_\mathcal{C} = \infty\} = \{\tau = \infty\}$ and we have made no restrictions on $\tilde{P}(\tau_\mathcal{I} = \infty)$ and $\tilde{P}(\tau_\mathcal{C} = \infty)$, both entities may not default at all. So, we allow for $\tilde{P}(\tau = \infty) \in [0,1]$.

Remark 3.1. The event $\{\tau_{\mathcal{I}} = \tau_{\mathcal{C}}, \tau < \infty\}$ of simultaneous default is a null set if $\tau_{\mathcal{I}}$ and $\tau_{\mathcal{C}}$ are $(\mathscr{F}_t)_{t\in[0,T]}$-conditionally independent under $\tilde{P}$. Indeed, in this case Lemma 2.5 gives

$$\tilde{P}(\tau_{\mathcal{I}} = \tau_{\mathcal{C}}, \tau < \infty) = \tilde{P}((\tau_{\mathcal{I}}, \tau_{\mathcal{C}}) \in \Delta) = \tilde{E}\bigg[\int_{[0,T]} K(\cdot, \{t\})\, L(\cdot, dt)\bigg] = 0 \tag{3.3}$$

for any two respective regular conditional probabilities K and L of $\tau_{\mathcal{I}}$ and $\tau_{\mathcal{C}}$ under $\tilde{P}$ given $\mathscr{F}_T$, where $\Delta := \{(s,t) \in [0,T]^2 \,|\, s = t\}$. Thereby, we note that

$$K(\omega, \{t\}) = 0 \quad \text{for all } (\omega, t) \in N^c \times [0,T]$$

and some null set $N \in \mathscr{F}_T$, since $\tilde{P}(\tau_{\mathcal{I}} = t|\mathscr{F}_t) = 0$ a.s. for all $t \in [0,T]$ and $\mathscr{B}([0,T]\cup\{\infty\})$ is countably generated. This justifies that the expectation in (3.3) vanishes.

Next, for any $(\mathscr{F}_t)_{t\in[0,T]}$-progressively measurable process γ with integrable paths we introduce an $]0,\infty[$-valued function $D(\gamma)$ on $[0,T]^2 \times \Omega$ by

$$D_{s,t}(\gamma) := \exp\bigg(-\int_s^t \gamma_{\tilde{s}}\, d\tilde{s}\bigg), \quad \text{if } s \leq t,$$

and $D_{s,t}(\gamma) := 1$, otherwise. Then the function $[0,T]^2 \to]0,\infty[$, $(s,t) \mapsto D_{s,t}(\gamma)(\omega)$ is continuous for any $\omega \in \Omega$ and $D_{s,t}(\gamma)$ is $\mathscr{F}_t$-measurable for all $s, t \in [0,T]$. Moreover, $D(\gamma)$ is bounded as soon as γ is bounded from below.

Let r be an $(\mathscr{F}_t)_{t\in[0,T]}$-progressively measurable process with integrable paths that represents the *instantaneous risk-free interest rate*. Then $D_{s,t}(r)$ is the *discount factor* from time $s \in [0,T]$ to $t \in [s,T]$. Put differently, $D_{s,t}(r)$ specifies the required amount to invest risk-free at time s, in order to receive 1 unit of cash at time t.

We let $\tilde{P}$ be a *local martingale measure* after a time $t_0 \in [0,T]$ in the sense that any discounted price process of a traded non-dividend-paying risky asset is an $(\tilde{\mathscr{F}}_t)_{t\in[t_0,T]}$-local martingale. That is, there is a non-empty set of processes $\tilde{U} \in \tilde{\mathscr{S}}$, representing the price processes of all such assets, for which $[t_0,T] \times \Omega \to \mathbb{R}$, $(t,\omega) \mapsto D_{0,t}(r)(\omega)\tilde{U}_t(\omega)$ is an $(\tilde{\mathscr{F}}_t)_{t\in[t_0,T]}$-local martingale under $\tilde{P}$. At all times, $\tilde{P}$ is ought to be equivalent to P.

Given the available market information, we will derive an equation for the *value process*, denoted by $\tilde{\mathscr{V}} \in \tilde{\mathscr{S}}$, of a trading strategy that hedges the contract between $\mathcal{I}$ and $\mathcal{C}$ under $\tilde{P}$ that leads to no arbitrage.

In the end, however, we seek a valuation that does not involve any knowledge of the default of any of the two parties, and the *valuation equation* for $\tilde{\mathscr{V}}$ includes quantities that merely depend on its pre-default part in the following sense.

As introduced in (2.2), let $G(\sigma)$ be an $(\mathscr{F}_t)_{t\in[0,T]}$-survival process under $\tilde{P}$ of a random variable σ with values in $[0,T]\cup\{\infty\}$, which is an $[0,1]$-valued $(\mathscr{F}_t)_{t\in[0,T]}$-supermartingale under $\tilde{P}$ such that

$$\tilde{P}(\sigma > t|\mathscr{F}_t) = G_t(\sigma) \quad \text{a.s. for all } t \in [0,T].$$

Let us call a process $\tilde{X}$ *integrable up to time* τ if $[0,T] \times \Omega \to \mathbb{R}$, $(t,\omega) \mapsto \tilde{X}_t(\omega)\mathbb{1}_{\{\tau>t\}}(\omega)$ is integrable. By Corollary 2.2, this property is satisfied by $\tilde{X} \in \tilde{\mathscr{S}}$ if and only if there is $X \in \mathscr{S}$ such that $XG(\tau)$ is integrable and $X_s = \tilde{X}_s$ a.s. on $\{\tau > s\}$ for each $s \in [0,T]$. In this case,

$$X_s G_s(\tau) = \tilde{E}[\tilde{X}_s \mathbb{1}_{\{\tau>s\}}|\mathscr{F}_s] \quad \text{a.s. for all } s \in [0,T] \tag{3.4}$$

and we shall call X a *pre-default version* of $\tilde{X}$. If in addition $G_s(\tau) > 0$ a.s. for every $s \in [0,T]$, which implies that the probability that neither $\mathcal{I}$ nor $\mathcal{C}$ defaults at any time is positive, then X is unique up to a modification.

In this spirit, we will introduce valuation based on default-free information only and analyse any *pre-default value process* $\mathscr{V}$ defined as pre-default version of $\tilde{\mathscr{V}}$, which in turn should be integrable up to time τ.

Remark 3.2. Any $(\mathscr{F}_t)_{t\in[t_0,T]}$-martingale X under $\tilde{P}$, in the local or standard sense, also satisfies the respective $(\tilde{\mathscr{F}}_t)_{t\in[t_0,T]}$-martingale property if for both $i \in \{\mathcal{I}, \mathcal{C}\}$ we have

$$G_s(\tau_i) = \tilde{P}(\tau_i > s|\mathscr{F}_t) \quad \text{a.s. for any } s,t \in [0,T] \text{ with } s \leq t \tag{3.5}$$

and $\tau_\mathcal{I}$, $\tau_\mathcal{C}$ are $(\mathscr{F}_t)_{t\in[0,T]}$-conditionally independent under $\tilde{P}$. Due to Lemma 2.7, these conditions are met in Example 3.9, which we will consider after analysing the model.

3.2 Incorporation of all relevant cash flows, costs and benefits

Let us summarise all cash flows, costs and benefits that may impact the value of the contract between $\mathcal{I}$ and $\mathcal{C}$. These quantities are the *contractual derivative cash flows* (3.6), the *costs and benefits of a collateral account* (3.7), the *funding costs and benefits* (3.8), the *repo costs and benefits associated to the hedging account* (3.9) and the *cash flows arising on the default of one of the two parties* (3.10).

Despite the contractual cash flows, all remaining quantities are allowed to depend on the value process $\tilde{\mathscr{V}}$ or its pre-default version $\mathscr{V}$. For a mathematical description we define a *time-dependent random functional* on a set $\mathscr{D}$ in $\tilde{\mathscr{S}}$ to be a function

$$F : [0,T] \times \Omega \times \mathscr{D} \to \mathbb{R}, \quad (t,\omega,X) \mapsto F_t(X)(\omega)$$

for which $F(X) : [0,T] \times \Omega \to \mathbb{R}$, $(t,\omega) \mapsto F_t(X)(\omega)$ is a process for every $X \in \mathscr{D}$. If there is a filtration $(\mathscr{G}_t)_{t\in[0,T]}$ of $\mathscr{F}$ to which $F(X)$ is adapted for all $X \in \mathscr{D}$, then we will refer to an $(\mathscr{G}_t)_{t\in[0,T]}$-time-dependent random functional.

(1) **The contractual derivative cash flows** with $\mathcal{C}$ are supposed to depend on a payoff functional and a dividend-paying risky asset that is influenced by its variance, or squared volatility, in an extended sense.

- The *price of the risky asset*, its *quasi variance* and its *instantaneous dividend rate* are modelled by two $\mathbb{R}_+$-valued càdlàg processes S and V in $\mathscr{S}$ and some process π that is $(\mathscr{F}_t)_{t\in[0,T]}$-progressively measurable and admits integrable paths, respectively.

- The $\mathbb{R}_+$-valued Borel measurable functional Φ defined on the closed set of all paths in $D([0,T]) \times D([0,T])$ with non-negative entries represents the *payoff functional*.

- The *contractual cash flows* consist of the amount $\Phi(S,V)$ paid at maturity and dividends according to the rate π. The continuous process ${}_{\mathrm{con}}\mathrm{CF}$ representing the discounted future cash flows at any time point is given by

$${}_{\mathrm{con}}\mathrm{CF}_s := D_{s,T}(r)\Phi(S,V)\mathbb{1}_{\{\tau>T\}} + \int_s^{T\wedge\tau} D_{s,t}(r)\pi_t\,dt. \tag{3.6}$$

(2) **The costs and benefits of a collateral account** arising from the collateralisation procedure to mitigate the default risk, subject to the collateral remuneration rate.

- Namely, the collateral serves as guarantee in case of default and the party receiving it will have to remunerate it at a certain interest rate, called the *collateral rate*, determined by the contract. We assume that the assets received as collateral can be re-hypotecated and do not have to be kept segregated.

- For an $(\mathscr{F}_t)_{t\in[0,T]}$-time-dependent random functional C on $\mathscr{S}$ the process $C(\mathscr{V})$, required to be càglàd, models the *cash flows of the collateral procedure.*

- The *collateral rates of each party* are represented by two $(\mathscr{F}_t)_{t\in[0,T]}$-progressively measurable processes ${}_+c$ and ${}_-c$ with integrable paths, respectively.

- So, $\mathcal{I}$ is a collateral receiver remunerating the assets at the rate ${}_+c_t$ on $\{C_t(\mathscr{V}) > 0\}$ and a collateral provider investing at the rate ${}_-c_t$ on $\{C_t(\mathscr{V}) < 0\}$ for all $t \in [0,T]$.

- By means of the $(\mathscr{F}_t)_{t\in[0,T]}$-time-dependent random functional c on $\mathscr{S}$ given by

$$c_t(X) := {}_+c_t \mathbb{1}_{]0,\infty[}(C_t(X)) + {}_-c_t \mathbb{1}_{]-\infty,0[}(C_t(X)),$$

the process $c(\mathscr{V})$ represents the *respective collateral rate*.

- We define a time-dependent random functional ${}_{\mathrm{col}}\mathrm{C}$ on the set of all $X \in \mathscr{S}$ for which $C(X)$ is càglàd by

$$ {}_{\mathrm{col}}\mathrm{C}_s(X) := \int_s^{T\wedge\tau} D_{s,t}(r)(c_t(X) - r_t)C_t(X)\,dt. \tag{3.7}$$

Then the continuous process ${}_{\mathrm{col}}\mathrm{C}(\mathscr{V})$ stands for the time evolution of the discounted future *collateral costs and benefits*.

(3) **The costs and benefits of a funding account** that may accrue, as $\mathcal{I}$ is supposed to have access to an account for borrowing or investing money at two respective risk-free interest rates.

- Given an $(\tilde{\mathscr{F}}_t)_{t\in[0,T]}$-time-dependent random functional $\tilde{F}$ on $\tilde{\mathscr{S}}$, the process $\tilde{F}(\tilde{\mathscr{V}})$, supposed to be càglàd, stands for the *funding amount*.

- The *interest rates for borrowing and lending* are given by the $(\tilde{\mathscr{F}}_t)_{t\in[0,T]}$-progressively measurable processes ${}_+\tilde{f}$ and ${}_-\tilde{f}$ with integrable paths, respectively.

- Hence, $\mathcal{I}$ is borrowing the amount $\tilde{F}_t(\tilde{\mathscr{V}})$ at the interest rate ${}_+\tilde{f}_t$ on $\{\tilde{F}_t(\tilde{\mathscr{V}}) > 0\}$ and she is lending $-\tilde{F}_t(\tilde{\mathscr{V}})$ at the rate ${}_-\tilde{f}_t$ on $\{\tilde{F}_t(\tilde{\mathscr{V}}) < 0\}$ for all $t \in [0,T]$.

- By using the $(\tilde{\mathscr{F}}_t)_{t\in[0,T]}$-time-dependent random functional $\tilde{f}$ on $\tilde{\mathscr{S}}$ defined via

$$\tilde{f}_t(\tilde{X}) := {}_+\tilde{f}_t \mathbb{1}_{]0,\infty[}(\tilde{F}_t(\tilde{X})) + {}_-\tilde{f}_t \mathbb{1}_{]-\infty,0[}(\tilde{F}_t(\tilde{X})),$$

the process $\tilde{f}(\tilde{\mathscr{V}})$ yields the *respective funding rate*.

- We introduce a time-dependent random functional ${}_{\mathrm{fun}}\mathrm{C}$ on the set of all $\tilde{X} \in \tilde{\mathscr{S}}$ for which $\tilde{F}(\tilde{X})$ is càglàd by

$$ {}_{\mathrm{fun}}\mathrm{C}_s(\tilde{X}) := \int_s^{T\wedge\tau} D_{s,t}(r)(\tilde{f}_t(\tilde{X}) - r_t)\tilde{F}_t(\tilde{X})\,dt. \tag{3.8}$$

Then the continuous process ${}_{\mathrm{fun}}\mathrm{C}(\tilde{\mathscr{V}})$ represents the temporal development of the present value of the *funding costs and benefits*.

(4) As $\mathcal{I}$ may be a bank $\mathcal{B}$, we assume that she may enter repurchase agreements to hedge its exposure. For this reason, **the repo costs and the benefits that result from hedging the derivative** should be taken into account.

- For an $(\tilde{\mathscr{F}}_t)_{t\in[0,T]}$-time-dependent random functional $\tilde{H}$ on $\tilde{\mathscr{S}}$ the process $\tilde{H}(\tilde{\mathscr{V}})$, assumed to be càglàd, measures the *value of the risky asset position that $\mathcal{I}$ has via the repo*.

- The two *repo rates* are given by $(\tilde{\mathscr{F}}_t)_{t\in[0,T]}$-progressively measurable processes ${}_+\tilde{h}$ and ${}_-\tilde{h}$ with integrable paths.

- Thus, $\mathcal{I}$ borrows a risky asset with the repo rate ${}_+\tilde{h}_t$ on $\{\tilde{H}_t(\tilde{\mathscr{V}}) > 0\}$ and lends a risky asset with the rate ${}_-\tilde{h}_t$ on $\{\tilde{H}_t(\tilde{\mathscr{V}}) < 0\}$ for each $t \in [0, T]$.

- We let the $(\tilde{\mathscr{F}}_t)_{t\in[0,T]}$-time-dependent random functional $\tilde{h}$ on $\tilde{\mathscr{S}}$ be given by

$$\tilde{h}_t(\tilde{X}) := {}_+\tilde{h}_t \mathbb{1}_{]0,\infty[}(\tilde{H}_t(\tilde{X})) + {}_-\tilde{h}_t \mathbb{1}_{]-\infty,0[}(\tilde{H}_t(\tilde{X})),$$

  which yields $\tilde{h}(\tilde{\mathscr{V}})$ as *respective repo rate.*

- Thereby, we suppose that $\mathcal{I}$ continuously rolls over repo contracts and that at each point $t \in [0, T]$ *she receives in the repo the exact value of the assets she is lending.* Thus, the gain of the repo position is given by the growth of the assets that are being repoed minus $\tilde{h}_t(\tilde{\mathscr{V}})(-\tilde{H}_t(\tilde{\mathscr{V}}))$, the repo rate times the amount of cash received.

- In this context, let the time-dependent random functional ${}_{\mathrm{hed}}\mathrm{C}$ on the set of all $\tilde{X} \in \tilde{\mathscr{S}}$ for which $\tilde{H}(\tilde{X})$ is càglàd be defined via

$$ {}_{\mathrm{hed}}\mathrm{C}_s(\tilde{X}) := \int_s^{T\wedge\tau} D_{s,t}(r)(r_t - \tilde{h}_t(\tilde{X}))\tilde{H}_t(\tilde{X})\, dt. \tag{3.9}$$

  Then the continuous process ${}_{\mathrm{hed}}\mathrm{C}(\tilde{\mathscr{V}})$ stands for the present value development of the *hedging costs and benefits.*

(5) **The cash flows arising on the default of one of the two parties** that can be computed with the *residual value of the claim*, the *net exposure*, the *losses given default* and the *funding amount.*

- For an $(\mathscr{F}_t)_{t\in[0,T]}$-time-dependent random functional ε on $\mathscr{S}$ the process $\varepsilon(\mathscr{V})$, which is ought to be càglàd, models the time evolution of the *close-out value.*

- We interpret $\varepsilon_\tau(\mathscr{V})$ as the *residual value of the claim* at the time τ of a party to default first on $\{\tau < \infty\}$, since $\tau_{\mathcal{I}} \neq \tau_{\mathcal{C}}$ a.s. on this event.

- On $\{\tau_{\mathcal{I}} > \tau_{\mathcal{C}}\}$ we specify that if the *net exposure* $(\varepsilon_\tau - C_\tau)(\mathscr{V})$ at the time of default is non-positive, then $\mathcal{I}$ is a net debtor and repays $\varepsilon_\tau(\mathscr{V})$ to $\mathcal{C}$.

- If instead $(\varepsilon_\tau - C_\tau)(\mathscr{V}) > 0$, then $\mathcal{I}$ is a net creditor and recovers a fraction $1 - \mathrm{LGD}_{\mathcal{I}}$ of its credits, in which case it receives $C_\tau(\mathscr{V}) + (1 - \mathrm{LGD}_{\mathcal{C}})(\varepsilon_\tau - C_\tau)(\mathscr{V})$.

- We implicitly assume that the loss fractions $\mathrm{LGD}_{\mathcal{I}}, \mathrm{LGD}_{\mathcal{C}} \in [0, 1]$, which denote the *losses given defaults* of $\mathcal{I}$ and $\mathcal{C}$, respectively, are deterministic exogenous quantities.

- The case in which $\mathcal{I}$ is a bank and defaults before $\mathcal{C}$ is symmetrical. If, however, $\mathcal{I} \neq \mathcal{B}$, then merely $\varepsilon_\tau(\mathscr{V})$ is being considered on $\{\tau_{\mathcal{I}} < \tau_{\mathcal{C}}\}$.

- We define a time-dependent random functional ${}_{\mathrm{def,c}}\mathrm{CF}$ on the set of all $X \in \mathscr{S}$ for which $\varepsilon(X)$ and $C(X)$ are càglàd via

$$\begin{aligned} {}_{\mathrm{def,c}}\mathrm{CF}_s(X) := {} & D_{s,\tau}(r)\big(\varepsilon_\tau(X) - \mathrm{LGD}_{\mathcal{C}}(\varepsilon_\tau - C_\tau)^+(X)\mathbb{1}_{\{\tau_{\mathcal{I}} > \tau_{\mathcal{C}}\}}\big) \\ & + D_{s,\tau}(r)\mathrm{LGD}_{\mathcal{I}}(\varepsilon_\tau - C_\tau)^-(X)\mathbb{1}_{\{\mathcal{I}=\mathcal{B},\, \tau_{\mathcal{I}} < \tau_{\mathcal{C}}\}} \end{aligned}$$

  on $\{s < \tau < T\}$ and ${}_{\mathrm{def,c}}\mathrm{CF}_s(X) := 0$ on the complement of this set. Then, according to our reasoning, the *discounted future cash flows on default due to the contract* can be modelled by the càdlàg process ${}_{\mathrm{def,c}}\mathrm{CF}(\mathscr{V})$.

- As we suppose that if $\mathcal{I}$ is a bank and has a cash surplus, then it may invest into risk-free assets, we also consider the *cash flows on the bank's default due to funding.*

- To this end, let the time-dependent random functional ${}_{\mathrm{def,f}}\mathrm{CF}$ on the set of all $\tilde{X} \in \tilde{\mathscr{S}}$ for which $\tilde{F}(\tilde{X})$ is càglàd be given by

$$ {}_{\mathrm{def,f}}\mathrm{CF}_s(\tilde{X}) := D_{s,\tau}(r)\mathrm{LGD}_{\mathcal{I}}\tilde{F}^+_\tau(\tilde{X})\mathbb{1}_{\{\mathcal{I}=\mathcal{B},\,\tau_{\mathcal{I}}<\tau_{\mathcal{C}}\}} $$

  on $\{s < \tau < T\}$ and ${}_{\mathrm{def,f}}\mathrm{CF}_s(\tilde{X}) := 0$ on its complement. Then the càdlàg process ${}_{\mathrm{def,f}}\mathrm{CF}(\tilde{\mathscr{V}})$ yields the time evolution of the corresponding net present value.

- Finally, for every $(X, \tilde{X}) \in \mathscr{S} \times \tilde{\mathscr{S}}$ for which $C(X)$, $\varepsilon(X)$ and $\tilde{F}(\tilde{X})$ are càglàd we set

$$ {}_{\mathrm{def}}\mathrm{CF}_s(X, \tilde{X}) := {}_{\mathrm{def,c}}\mathrm{CF}_s(X) + {}_{\mathrm{def,f}}\mathrm{CF}_s(\tilde{X}). \tag{3.10} $$

  Then the process ${}_{\mathrm{def}}\mathrm{CF}(\mathscr{V}, \tilde{\mathscr{V}})$ sums up both sources of default risk.

3.3 The pre-default valuation equation

For the valuation of the derivative contract let us first ensure the integrability of the net present values of all the cash flows, costs and benefits generally given by (3.6)-(3.10). Throughout this section, $(\Omega, \mathscr{F}, \tilde{P})$ serves as underlying probability space.

Let $\tilde{\mathscr{L}}(r,\tau)$ be the linear space of all random variables X for which $D_{s,T}(r)|X|\mathbb{1}_{\{\tau>T\}}$ is $\tilde{P}$-integrable for any $s \in [0,T]$ and $\tilde{\mathscr{S}}(r,\tau)$ be the linear space of all measurable processes X satisfying

$$ \tilde{E}\bigg[\int_s^{T\wedge\tau} D_{s,t}(r)|X_t|\,dt\bigg] < \infty \quad \text{for all } s \in [0,T[. $$

Furthermore, by $\tilde{\mathscr{D}}(r,\tau)$ we denote the linear space of all càglàd processes X such that $\sup_{t\in]s,T[} D_{s,t}(r)|X_t|\mathbb{1}_{\{s<\tau<T\}}$ is $\tilde{P}$-integrable for each $s \in [0,T[$ and we set

$$ \tilde{\mathscr{L}}(r) := \tilde{\mathscr{L}}(r,\infty) \quad \text{and} \quad \tilde{\mathscr{S}}(r) := \tilde{\mathscr{S}}(r,\infty). $$

Remark 3.3. For any two $\mathbb{R}_+$-valued random variables X and $\tilde{X}$ that are measurable with respect to $\mathscr{F}_T$ and $\tilde{\mathscr{F}}_T$ such that $X = \tilde{X}$ a.s. on $\{\tau > T\}$ Corollary 2.2 implies

$$ \tilde{E}[D_{s,T}(r)\tilde{X}\mathbb{1}_{\{\tau>T\}}] = \tilde{E}[D_{s,T}(r)XG_T(\tau)] \quad \text{for any } s \in [0,T]. $$

Thus, $\tilde{X} \in \tilde{\mathscr{L}}(r,\tau) \Leftrightarrow XG_T(\tau) \in \tilde{\mathscr{L}}(r)$. Further, if $G(\tau)$ is measurable and now X and $\tilde{X}$ denote two $\mathbb{R}_+$-valued processes that are progressively measurable relative to $(\mathscr{F}_t)_{t\in[0,T]}$ and $(\tilde{\mathscr{F}}_t)_{t\in[0,T]}$, respectively, then

$$ \tilde{E}\bigg[\int_s^{T\wedge\tau} D_{s,t}(r)\tilde{X}_t\,dt\bigg] = \tilde{E}\bigg[\int_s^{T} D_{s,t}(r)X_tG_t(\tau)\,dt\bigg] $$

as soon as $X_t = \tilde{X}_t$ a.s. on $\{\tau > t\}$ for all $t \in [0,T]$, according to Lemma 2.3. This in turn shows that $\tilde{X} \in \tilde{\mathscr{S}}(r,\tau) \Leftrightarrow XG(\tau) \in \tilde{\mathscr{S}}(r)$.

Based on our definitions, the process ${}_{\mathrm{con}}\mathrm{CF}$ given by (3.6) that models the discounted contractual derivative cash flows is integrable if our first model assumption holds:

(M.1) The amount $\Phi(S,V)$ paid at maturity and the dividend rate π lie in $\tilde{\mathscr{L}}(r,\tau)$ and $\tilde{\mathscr{S}}(r,\tau)$, respectively.

By Remark 3.3, if $G(\tau)$ were measurable, then, equivalently, we could have asked for $\Phi(S,V)G_T(\tau) \in \tilde{\mathscr{L}}(r)$ and $\pi G(\tau) \in \tilde{\mathscr{S}}(r)$. To consider the remaining quantities, let $\tilde{\mathscr{V}} \in \tilde{\mathscr{S}}$ be integrable up to time τ and $\mathscr{V}$ be a pre-default version of it.

We suppose that the collateral process, the funding amount, the hedging process and the close-out value possess càglàd paths or in short,

$$C(\mathscr{V}),\ \tilde{F}(\tilde{\mathscr{V}}),\ \tilde{H}(\tilde{\mathscr{V}}) \text{ and } \varepsilon(\mathscr{V}) \text{ are càglàd.} \tag{3.11}$$

Then the processes ${}_{\text{col}}\text{C}(\mathscr{V})$, ${}_{\text{fun}}\text{C}(\tilde{\mathscr{V}})$ and ${}_{\text{hed}}\text{C}(\tilde{\mathscr{V}})$, introduced in (3.7)-(3.9), that model the collateral, funding and hedging costs and benefits, respectively, are integrable if

$$(c(\mathscr{V}) - r)C(\mathscr{V}),\ (\tilde{f}(\tilde{\mathscr{V}}) - r)\tilde{F}(\tilde{\mathscr{V}}),\ (r - \tilde{h}(\tilde{\mathscr{V}}))\tilde{H}(\tilde{\mathscr{V}}) \text{ lie in } \tilde{\mathscr{S}}(r,\tau). \tag{3.12}$$

For the integrability of ${}_{\text{def}}\text{CF}(\mathscr{V},\tilde{\mathscr{V}})$, given by (3.10) and modelling the cash flows on the default of one of the two parties, it suffices that the respective cash flows appearing on the default of $\mathcal{I}$ and $\mathcal{C}$ are elements of $\tilde{\mathscr{D}}(r,\tau)$. Namely,

$$\begin{aligned}&(\varepsilon(\mathscr{V}) + \text{LGD}_{\mathcal{I}}((\varepsilon - C)^-(\mathscr{V}) + \tilde{F}^+(\tilde{\mathscr{V}}))\mathbb{1}_{\{\mathcal{I}=\mathcal{B}\}})\mathbb{1}_{\{\tau_{\mathcal{I}}<\tau_{\mathcal{C}}\}}\\ &\text{and } (\varepsilon - \text{LGD}_{\mathcal{C}}(\varepsilon - C)^+)(\mathscr{V})\mathbb{1}_{\{\tau_{\mathcal{I}}>\tau_{\mathcal{C}}\}} \text{ belong to } \tilde{\mathscr{D}}(r,\tau).\end{aligned} \tag{3.13}$$

In fact, as condition (3.2) states that $\tau_{\mathcal{I}} \neq \tau_{\mathcal{C}}$ a.s. on $\{\tau < \infty\}$, we readily check that the random variable $|{}_{\text{def}}\text{CF}_s(\mathscr{V},\tilde{\mathscr{V}})|$ is bounded by

$$\begin{aligned}&\sup_{t\in]s,T[} D_{s,t}(r)|\varepsilon_t(\mathscr{V}) + \text{LGD}_{\mathcal{I}}((\varepsilon_t - C_t)^-(\mathscr{V}) + \tilde{F}_t^+(\tilde{\mathscr{V}}))\mathbb{1}_{\{\mathcal{I}=\mathcal{B}\}}|\mathbb{1}_{\{\tau_{\mathcal{I}}<\tau_{\mathcal{C}}\}}\\ &\quad + \sup_{t\in]s,T[} D_{s,t}(r)|\varepsilon_t - \text{LGD}_{\mathcal{C}}(\varepsilon_t - C_t)^+|(\mathscr{V})\mathbb{1}_{\{\tau_{\mathcal{I}}>\tau_{\mathcal{C}}\}}\end{aligned}$$

a.s. on $\{s < \tau < T\}$ for all $s \in [0,T[$, which entails the asserted integrability. Let us now suppose that (M.1) holds. For $\tilde{\mathscr{V}}$ to be the value process of a hedging strategy of the contract under $\tilde{P}$, we require that (3.11)-(3.13) be satisfied.

Further, we stipulate that $\tilde{\mathscr{V}}_s$ agrees with the conditional expectation of the sum of the net present values of all cash flows, costs and benefits relative to the current available market information under $\tilde{P}$ for each $s \in [0,T]$. Namely,

$$\tilde{\mathscr{V}}_s = \tilde{E}\big[{}_{\text{con}}\text{CF}_s - {}_{\text{col}}\text{C}_s(\mathscr{V}) - {}_{\text{fun}}\text{C}_s(\tilde{\mathscr{V}}) - {}_{\text{hed}}\text{C}_s(\tilde{\mathscr{V}}) + {}_{\text{def}}\text{CF}_s(\mathscr{V},\tilde{\mathscr{V}})|\tilde{\mathscr{F}}_s\big] \quad \text{a.s.} \tag{3.14}$$

for any $s \in [0,T]$ and from our considerations we infer that $\tilde{\mathscr{V}}$ is necessarily $\tilde{P}$-integrable. In particular, the terminal value condition $\tilde{\mathscr{V}}_T = {}_{\text{con}}\text{CF}_T = \Phi(S,V)$ a.s. must hold.

This implicit conditional representation refines the *valuation equation* (1) in [12], which is build on the valuation problems in [30] and [31]. As immediate consequence of (3.4), the pre-default version $\mathscr{V}$ of the value process $\tilde{\mathscr{V}}$ satisfies

$$\mathscr{V}_s G_s(\tau) = \tilde{E}\big[{}_{\text{con}}\text{CF}_s - {}_{\text{col}}\text{C}_s(\mathscr{V}) - {}_{\text{fun}}\text{C}_s(\tilde{\mathscr{V}}) - {}_{\text{hed}}\text{C}_s(\tilde{\mathscr{V}}) + {}_{\text{def}}\text{CF}_s(\mathscr{V},\tilde{\mathscr{V}})|\mathscr{F}_s\big] \quad \text{a.s.} \tag{3.15}$$

for all $s \in [0,T]$, as the quantities ${}_{\text{con}}\text{CF}_s$, ${}_{\text{col}}\text{C}_s(\mathscr{V})$, ${}_{\text{fun}}\text{C}_s(\tilde{\mathscr{V}})$, ${}_{\text{hed}}\text{C}_s(\tilde{\mathscr{V}})$ and ${}_{\text{def}}\text{CF}_s(\mathscr{V},\tilde{\mathscr{V}})$ vanish on $\{\tau \leq s\}$. Thus, to derive a valuation equation for $\mathscr{V}$ restricted to default-free information, we will replace the $\mathscr{F}_T \vee \mathscr{H}_T^{\tau_{\mathcal{I}}} \vee \mathscr{H}_T^{\tau_{\mathcal{C}}}$-measurable random variables

$${}_{\text{con}}\text{CF}_s,\quad {}_{\text{col}}\text{C}_s(\mathscr{V}),\quad {}_{\text{fun}}\text{C}_s(\tilde{\mathscr{V}}),\quad {}_{\text{hed}}\text{C}_s(\tilde{\mathscr{V}}) \quad \text{and} \quad {}_{\text{def}}\text{CF}_s(\mathscr{V},\tilde{\mathscr{V}})$$

within the conditional expectation in (3.15) by $\mathscr{F}_T$-measurable ones that may merely depend on $\mathscr{V}$. For this purpose, we will use the probabilistic results from Section 2.2 and require a set of model assumptions:

(M.2) $G(\tau_{\mathcal{I}})$ and $G(\tau_{\mathcal{C}})$ are continuous and of finite variation, $\tau_{\mathcal{I}}$ and $\tau_{\mathcal{C}}$ are conditionally independent relative to $(\mathscr{F}_t)_{t\in[0,T]}$ under $\tilde{P}$ and $G(\tau) = G(\tau_{\mathcal{I}})G(\tau_{\mathcal{C}})$.

(M.3) There are two $(\mathscr{F}_t)_{t\in[0,T]}$-time-dependent random functionals F and H on $\mathscr{S}$ such that $F(X)$ and $H(X)$ serve as pre-default versions of $\tilde{F}(\tilde{X})$ and $\tilde{H}(\tilde{X})$, respectively, for any $\tilde{X} \in \tilde{\mathscr{S}}$ that is integrable up to time τ with pre-default version X.

(M.4) The funding rates ${}_+\tilde{f}$, ${}_-\tilde{f}$ and the hedging rates ${}_+\tilde{h}$, ${}_-\tilde{h}$ are integrable up to time τ and admit $(\mathscr{F}_t)_{t\in[0,T]}$-progressively measurable pre-default versions ${}_+f$, ${}_-f$ and ${}_+h$, ${}_-h$, respectively, with integrable paths.

Remark 3.4. The identity in (M.2) simply states that $G(\tau)$ and $G(\tau_{\mathcal{I}})G(\tau_{\mathcal{C}})$ are not only modifications of each other, but in fact equal. This ensures that all the paths of $G(\tau)$ are continuous and of finite variation.

Under (M.3) and (M.4), we may define two time-dependent random functionals f and h on $\mathscr{S}$ relative to $(\mathscr{F}_t)_{t\in[0,T]}$ by $f_t(X) := {}_+f_t\mathbb{1}_{]0,\infty[}(F_t(X)) + {}_-f_t\mathbb{1}_{]-\infty,0[}(F_t(X))$ and

$$h_t(X) := {}_+h_t\mathbb{1}_{]0,\infty[}\big(H_t(X)\big) + {}_-h_t\mathbb{1}_{]-\infty,0[}\big(H_t(X)\big).$$

Then $\tilde{f}(\tilde{X})$ and $\tilde{h}(\tilde{X})$ admit $f(X)$ and $h(X)$ as pre-default versions, respectively, for any $\tilde{X} \in \tilde{\mathscr{S}}$ that is integrable up to time τ with pre-default version X. Further, $f(X)$ and $h(X)$ are $(\mathscr{F}_t)_{t\in[0,T]}$-progressively measurable if $F(X)$ and $H(X)$ are.

Now let the pre-default funding amount $F(\mathscr{V})$ and the pre-default hedging process $H(\mathscr{V})$ be càglàd. In this case, (3.11) entails that the following path regularity condition for the pre-default version $\mathscr{V}$ of $\tilde{\mathscr{V}}$ holds:

(C.1) $C(\mathscr{V})$, $F(\mathscr{V})$, $H(\mathscr{V})$ and $\varepsilon(\mathscr{V})$ are càglàd.

Let in addition (M.2) be satisfied. Then Remark 3.3 shows that (3.12) is valid if and only if the following integrability condition is valid:

(C.2) The product of $G(\tau)$ with any of the processes $(c(\mathscr{V}) - r)C(\mathscr{V})$, $(f(\mathscr{V}) - r)F(\mathscr{V})$ and $(r - h(\mathscr{V}))H(\mathscr{V})$ belongs to $\tilde{\mathscr{S}}(r)$.

To handle the conditional expectation of ${}_{\mathrm{def}}\mathrm{CF}_s(\mathscr{V}, \tilde{\mathscr{V}})$ relative to $\mathscr{F}_s$ in (3.15) for any $s \in [0, T]$, we need another integrability condition that involves the variation process $V(\tau_i)$ of $G(\tau_i)$ for both $i \in \{\mathcal{I}, \mathcal{C}\}$. In this regard, we recall that $V(\tau_i) = 1 - G(\tau_i) \in [0, 1]$ if $G(\tau_i)$ is decreasing, as in Example 3.9 below.

(C.3) $\sup_{t\in]s,T[} D_{s,t}(r)|\varepsilon_t + \mathrm{LGD}_{\mathcal{I}}((\varepsilon_t - C_t)^- + F_t^+)\mathbb{1}_{\{\mathcal{I}=\mathcal{B}\}}|(\mathscr{V})G_t(\tau_{\mathcal{C}})(V_T(\tau_{\mathcal{I}}) - V_s(\tau_{\mathcal{I}}))$ and

$$\sup_{t\in]s,T[} D_{s,t}(r)|\varepsilon_t - \mathrm{LGD}_{\mathcal{C}}(\varepsilon_t - C_t)^+|(\mathscr{V})G_t(\tau_{\mathcal{I}})(V_T(\tau_{\mathcal{C}}) - V_s(\tau_{\mathcal{C}}))$$

are $\tilde{P}$-integrable for any $s \in [0, T[$.

For a clear and concise overview, let us summarise all appearing quantities by defining three $(\mathscr{F}_t)_{t\in[0,T]}$-time-dependent random functionals ${}_0\mathrm{B}$, ${}_{\mathcal{I}}\mathrm{B}$ and ${}_{\mathcal{C}}\mathrm{B}$ on $\mathscr{S}$ via

$$\begin{aligned}
{}_0\mathrm{B}_t(X) &:= \pi_t - (c_t(X) - r_t)C_t(X) - (f_t(X) - r_t)F_t(X) - (r_t - h_t(X))H_t(X),\\
{}_{\mathcal{I}}\mathrm{B}_t(X) &:= \varepsilon_t(X) + \mathrm{LGD}_{\mathcal{I}}((\varepsilon_t - C_t)^- + F_t^+)(X)\mathbb{1}_{\{\mathcal{I}=\mathcal{B}\}} \quad \text{and}\\
{}_{\mathcal{C}}\mathrm{B}_t(X) &:= \varepsilon_t(X) - \mathrm{LGD}_{\mathcal{C}}(\varepsilon_t - C_t)^+(X).
\end{aligned} \tag{3.16}$$

Then ${}_0\mathrm{B}(X)$ is $(\mathscr{F}_t)_{t\in[0,T]}$-progressively measurable for any $X \in \mathscr{S}$ as soon as $C(X)$, $F(X)$ and $H(X)$ are. Thus, (C.2) implies ${}_0\mathrm{B}(\mathscr{V})G(\tau) \in \tilde{\mathscr{S}}(r)$. Further, ${}_{\mathcal{I}}\mathrm{B}(X)$ and ${}_{\mathcal{C}}\mathrm{B}(X)$ are càglàd if $C(X)$, $F(X)$ and $\varepsilon(X)$ are. In this case, the two Riemann-Stieltjes integrals

$$\int_s^T D_{s,t}(r)|{}_{\mathcal{I}}\mathrm{B}_t(X)|G_t(\tau_{\mathcal{C}})\,dV_t(\tau_{\mathcal{I}}) \quad \text{and} \quad \int_s^T D_{s,t}(r)|{}_{\mathcal{C}}\mathrm{B}_t(X)|G_t(\tau_{\mathcal{I}})\,dV_t(\tau_{\mathcal{C}})$$

are finite for each $s \in [0, T[$, and for $X = \mathscr{V}$ each of these integrals is bounded by the respective random variable in (C.3), which ensures their $\tilde{P}$-integrability.

Consequently, we may introduce an $(\mathscr{F}_t)_{t\in[0,T]}$-time-dependent random functional A on the set of all $X \in \mathscr{S}$ for which $C(X)$, $F(X)$, $H(X)$ and $\varepsilon(X)$ are càglàd by

$$\begin{aligned} A_t(X) := \int_0^t {}_0\mathrm{B}_s(X)G_s(\tau)\,ds - \int_0^t {}_{\mathcal{I}}\mathrm{B}_s(X)G_s(\tau_{\mathcal{C}})\,dG_s(\tau_{\mathcal{I}}) \\ - \int_0^t {}_{\mathcal{C}}\mathrm{B}_s(X)G_s(\tau_{\mathcal{I}})\,dG_s(\tau_{\mathcal{C}}). \end{aligned} \tag{3.17}$$

We readily see that $A(X)$ is a continuous process of finite variation for each process X in its domain and the Riemann-Stieltjes integral $\int_s^T D_{s,t}(r)\,dA_t(\mathscr{V})$ is $\tilde{P}$-integrable for every $s \in [0, T]$ if (C.2) and (C.3) are valid.

Based on all these measurability, path regularity and integrability considerations, we may now rewrite the conditional expectations appearing in (3.15) as follows.

Proposition 3.5. *Let* (M.1)-(M.4) *hold and* $\tilde{\mathscr{V}} \in \tilde{\mathscr{S}}$ *be integrable with pre-default version* $\mathscr{V}$ *such that* $\tilde{F}(\tilde{\mathscr{V}})$, $\tilde{H}(\tilde{\mathscr{V}})$ *are càglàd and* (3.13) *and* (C.1)-(C.3) *hold. Then*

$$\begin{aligned} \tilde{E}\big[{}_{\mathrm{con}}\mathrm{CF}_s - {}_{\mathrm{col}}\mathrm{C}_s(\mathscr{V}) - {}_{\mathrm{fun}}\mathrm{C}_s(\tilde{\mathscr{V}}) - {}_{\mathrm{hed}}\mathrm{C}_s(\tilde{\mathscr{V}}) + {}_{\mathrm{def}}\mathrm{CF}_s(\mathscr{V}, \tilde{\mathscr{V}})\big|\mathscr{F}_s\big] \\ = \tilde{E}\bigg[D_{s,T}(r)\Phi(S,V)G_T(\tau) + \int_s^T D_{s,t}(r)\,dA_t(\mathscr{V})\,\bigg|\,\mathscr{F}_s\bigg] \end{aligned} \tag{3.18}$$

a.s. for each $s \in [0, T]$.

This result leads us to a valuation equation involving default-free information only. Namely, by a solution to the *pre-default valuation equation*

$$\mathscr{V}_s G_s(\tau) = \tilde{E}\bigg[D_{s,T}(r)\Phi(S,V)G_T(\tau) + \int_s^T D_{s,t}(r)\,dA_t(\mathscr{V})\,\bigg|\,\mathscr{F}_s\bigg] \quad \text{a.s.} \tag{VE}$$

for $s \in [t_0, T]$ we shall mean a process $\mathscr{V} \in \mathscr{S}$ such that the path regularity condition (C.1) and the two integrability conditions (C.2) and (C.3) hold and the almost sure identity in (VE) is satisfied for each $s \in [t_0, T]$.

In this case, $\mathscr{V}_s G_s(\tau)$ is necessarily integrable and $\mathscr{V}_T = \Phi(S, V)$ a.s. on $\{G_T(\tau) > 0\}$. Thus, we obtain a *martingale characterisation* for any such *pre-default value process*.

Proposition 3.6. *Assume that* (M.1)-(M.4) *are valid and* $\mathscr{V} \in \mathscr{S}$ *satisfies* (C.1)-(C.3). *Then the* $(\mathscr{F}_t)_{t\in[0,T]}$*-adapted continuous process*

$$[0, T] \times \Omega \to \mathbb{R}, \quad (t, \omega) \mapsto \int_0^t D_{0,s}(r)(\omega)\,dA_s(\mathscr{V})(\omega) \tag{3.19}$$

of finite variation is integrable. Moreover, $\mathscr{V}$ *solves* (VE) *if and only if* ${}_{\mathscr{V}}M \in \mathscr{S}$ *defined via*

$${}_{\mathscr{V}}M_t := D_{0,t}(r)\mathscr{V}_t G_t(\tau) + \int_0^t D_{0,s}(r)\,dA_s(\mathscr{V}) \tag{3.20}$$

is an $(\mathscr{F}_t)_{t\in[t_0,T]}$*-martingale under* $\tilde{P}$ *and* $\mathscr{V}_T = \Phi(S, V)$ *a.s. on* $\{G_T(\tau) > 0\}$.

For an *implicit backward stochastic integral representation of any pre-default value process* we assume until the end of this section that $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \tilde{P})$ satisfies the usual conditions.

Proposition 3.7. *Let* (M.1)-(M.4) *be valid and* (C.1)-(C.3) *hold for* $\mathscr{V} \in \mathscr{S}$. *Then* $\mathscr{V}G(\tau)$ *is a continuous* $(\mathscr{F}_t)_{t\in[t_0,T]}$*-semimartingale if and only if* ${}_{\mathscr{V}}M$ *is. In this case,*

$$\mathscr{V}_s G_s(\tau) = \mathscr{V}_T G_T(\tau) + \int_s^T \big(dA_t(\mathscr{V}) - r_t \mathscr{V}_t G_t(\tau)\,dt\big) - \int_s^T D_{0,t}(-r)\,d_{\mathscr{V}}M_t \tag{3.21}$$

for any $s \in [t_0, T]$ *a.s. If in addition* $G(\tau) > 0$, *then* ${}_{\mathscr{V}}M$ *is, up to indistinguishability, the unique continuous* $(\mathscr{F}_t)_{t\in[t_0,T]}$*-semimartingale satisfying*

$$\begin{aligned}\mathscr{V}_s = \mathscr{V}_T &+ \int_s^T \big({}_0\mathrm{B}_t(\mathscr{V}) - r_t\mathscr{V}_t\big)\,dt - \int_s^T \frac{{}_{\mathcal{I}}\mathrm{B}_t(\mathscr{V}) - \mathscr{V}_t}{G_t(\tau_{\mathcal{I}})}\,dG_t(\tau_{\mathcal{I}}) \\ &- \int_s^T \frac{{}_{\mathcal{C}}\mathrm{B}_t(\mathscr{V}) - \mathscr{V}_t}{G_t(\tau_{\mathcal{C}})}\,dG_t(\tau_{\mathcal{C}}) - \int_s^T \frac{D_{0,t}(-r)}{G_t(\tau)}\,d_{\mathscr{V}}M_t\end{aligned} \tag{3.22}$$

for all $s \in [t_0, T]$ *a.s. and* ${}_{\mathscr{V}}M_{t_1} = D_{0,t_1}(r)\mathscr{V}_{t_1}G_{t_1}(\tau) + \int_0^{t_1} D_{0,t}(r)\,dA_t(\mathscr{V})$ *a.s. for some* $t_1 \in [t_0, T]$.

As a consequence of the preceding two results, we are able to *characterise pre-default value processes that are semimartingales.*

Corollary 3.8. *Let* (M.1)-(M.4) *hold,* $G(\tau) > 0$, $\mathscr{V} \in \mathscr{S}$ *be continuous and* (C.1)-(C.3) *be valid. Then* $\mathscr{V}$ *is an* $(\mathscr{F}_t)_{t\in[t_0,T]}$*-semimartingale solving* (VE) *if and only if*

$$D_{0,t_0}(r)\mathscr{V}_{t_0}G_{t_0}(\tau) \text{ is } \tilde{P}\text{-integrable}, \quad \mathscr{V}_T = \Phi(S,V) \quad a.s.$$

and there is a continuous $(\mathscr{F}_t)_{t\in[t_0,T]}$*-martingale* M *that takes the role of* ${}_{\mathscr{V}}M$ *in* (3.22). *If this is the case, then* $M_t - M_{t_0} = {}_{\mathscr{V}}M_t - {}_{\mathscr{V}}M_{t_0}$ *for all* $t \in [t_0, T]$ *a.s.*

In addition to our four model assumptions let $G(\tau_{\mathcal{I}})$ and $G(\tau_{\mathcal{C}})$ be not only continuous and of finite variation but absolutely continuous and positive. Then the same holds for $G(\tau)$ and any continuous $\mathscr{V} \in \mathscr{S}$ satisfying (C.1)-(C.3) solves (VE) if and only if

$$\begin{aligned}\mathscr{V}_s = \tilde{E}&\bigg[D_{s,T}(r)\Phi(S,V)\frac{G_T(\tau)}{G_s(\tau)}\,\bigg|\,\mathscr{F}_s\bigg] \\ &+ \tilde{E}\bigg[\int_s^T D_{s,t}(r)\frac{G_t(\tau)}{G_s(\tau)}\bigg(\mathrm{B}_t(\mathscr{V}) + \bigg(r_t - \frac{\dot{G}_t(\tau)}{G_t(\tau)}\bigg)\mathscr{V}_t\bigg)\,dt\,\bigg|\,\mathscr{F}_s\bigg]\end{aligned} \tag{3.23}$$

for any $s \in [t_0, T]$ with the $(\mathscr{F}_t)_{t\in[0,T]}$-time-dependent random functional B defined on the set of all continuous $X \in \mathscr{S}$ for which $C(X)$, $F(X)$, $H(X)$ and $\varepsilon(X)$ are càglàd by

$$\mathrm{B}_t(X) := {}_0\mathrm{B}_t(X) - r_tX_t - \frac{\dot{G}_t(\tau_{\mathcal{I}})}{G_t(\tau_{\mathcal{I}})}({}_{\mathcal{I}}\mathrm{B}_t(X) - X_t) - \frac{\dot{G}_t(\tau_{\mathcal{C}})}{G_t(\tau_{\mathcal{C}})}({}_{\mathcal{C}}\mathrm{B}_t(X) - X_t). \tag{3.24}$$

Moreover, if $\mathscr{V}$, or equivalently, ${}_{\mathscr{V}}M$ is an $(\mathscr{F}_t)_{t\in[t_0,T]}$-semimartingale, then we may rewrite the implicit backward stochastic integral representation (3.22) in the form

$$\mathscr{V}_s = \mathscr{V}_T + \int_s^T \mathrm{B}_t(\mathscr{V})\,dt - \int_s^T \frac{D_{0,t}(-r)}{G_t(\tau)}\,d_{\mathscr{V}}M_t \quad \text{for all } s \in [t_0,T] \text{ a.s.} \tag{3.25}$$

Example 3.9. For both $i \in \{\mathcal{I}, \mathcal{C}\}$ let $\lambda^{(i)}$ be an $\mathbb{R}_+$-valued $(\mathscr{F}_t)_{t\in[0,T]}$-progressively measurable process with integrable paths such that every $\omega \in \Omega$ satisfies

$$\int_0^{t_\omega} \lambda_s^{(i)}(\omega)\,ds > 0 \quad \text{for some } t_\omega \in]0,T],$$

which holds if $\lambda^{(i)} > 0$, for instance. Further, let ξ_i be an $]0,\infty[$-valued $\tilde{\mathscr{F}}_0$-measurable random variable that is gamma distributed with shape $\alpha_i > 0$ and rate $\beta_i > 0$ such that

$$\tau_i = \inf\bigg\{t \in [0,T] \,\bigg|\, \int_0^t \lambda_s^{(i)}\,ds \geq \xi_i\bigg\}.$$

We suppose that $(\xi_{\mathcal{I}},\xi_{\mathcal{C}})$ is independent of $\mathscr{F}_T$ and $\xi_{\mathcal{I}}$ and $\xi_{\mathcal{C}}$ are independent. Then Lemma 2.7 and Example 2.12 show that (3.2) and (M.2) hold for $G(\tau) = G(\tau_{\mathcal{I}})G(\tau_{\mathcal{C}})$ and both $G(\tau_{\mathcal{I}})$ and $G(\tau_{\mathcal{C}})$ are absolutely continuous and positive. Moreover,

$$-\frac{\dot{G}_t(\tau_i)}{G_t(\tau_i)} = \frac{\beta_i^{\alpha_i}\lambda_t^{(i)}}{\gamma(\alpha_i,\beta_i\int_0^t\lambda_s^{(i)}\,ds)}\bigg(\int_0^t \lambda_s^{(i)}\,ds\bigg)^{\alpha_i-1}\exp\bigg(-\beta_i\int_0^t\lambda_s^{(i)}\,ds\bigg)$$

for a.e. $t \in [0,T]$ for both $i \in \{\mathcal{I},\mathcal{C}\}$, where γ is the upper incomplete gamma function, and this formula this reduces to $-\dot{G}(\tau_i)/G(\tau_i) = \beta_i\lambda^{(i)}$ a.e. when $\alpha_i = 1$. Thus, in the case $\alpha_{\mathcal{I}} = \alpha_{\mathcal{C}} = 1$ we have

$$\begin{aligned}\mathrm{B}_t(X) = {}_0\mathrm{B}_t(X) - r_tX_t + \big(\beta_{\mathcal{I}}\lambda_t^{(\mathcal{I})} + \beta_{\mathcal{C}}\lambda_t^{(\mathcal{C})}\big)\big(\varepsilon_t(X) - X_t\big)\\ + \beta_{\mathcal{I}}\lambda_t^{(\mathcal{I})}\mathrm{LGD}_{\mathcal{I}}((\varepsilon_t - C_t)^- + F_t^+)(X)\mathbb{1}_{\{\mathcal{I}=\mathcal{B}\}} - \beta_{\mathcal{C}}\lambda_t^{(\mathcal{C})}\mathrm{LGD}_{\mathcal{C}}(\varepsilon_t - C_t)^+(X)\end{aligned}$$

for a.e. $t \in [0,T]$ and any $X \in \mathscr{S}$ for which $C(X)$, $F(X)$, $H(X)$ and $\varepsilon(X)$ are càglàd. In particular, if $\xi_{\mathcal{I}}$ and $\xi_{\mathcal{C}}$ are exponentially distributed with mean one and the *financing hypothesis*

$$X_t = C_t(X) + F_t(X) \quad \text{for all } (t,X) \in [0,T]\times\mathscr{S} \tag{3.26}$$

holds, then (3.23) and (3.25) yield the respective identities (5) and (6) for the pre-default value process in [12], where $(\mathscr{F}_t)_{t\in[0,T]}$ is the augmented filtration of a standard Brownian motion and the martingale representation theorem may be applied.

4 A parabolic equation for the pre-default valuation

4.1 A general stochastic volatility model

In the sequel, let the filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in[0,T]},P)$ satisfy the usual conditions and suppose that there are two standard $(\mathscr{F}_t)_{t\in[0,T]}$-Brownian motions $\hat{W}$ and $\tilde{W}$ with covariation

$$\langle\hat{W},\tilde{W}\rangle_t = \int_0^t \rho(s)\,ds \quad \text{for all } t \in [0,T] \text{ a.s,}$$

where $\rho : [0,T] \to]-1,1[$ is a measurable function satisfying $\int_0^T (1-\rho(s)^2)^{-1}\,ds < \infty$. Let $b : [0,T] \to \mathbb{R}$ and $\zeta,\eta,\theta : [0,T]\times\mathbb{R} \to \mathbb{R}$ be measurable and consider the two-dimensional SDE starting at time $t_0 \in [0,T]$:

$$\begin{aligned} dS_t &= b(t)S_t\,dt + \theta(t,V_t)S_t\,d\hat{W}_t,\\ dV_t &= \zeta(t,V_t)\,dt + \eta(t,V_t)\,d\tilde{W}_t \end{aligned} \tag{4.1}$$

for $t \in [t_0,T]$. Given any weak solution (S,V), we will interpret S as *price process of the risky asset* and V as *quasi variance* or *quasi squared volatility process* influencing S via the function θ, which will satisfy the 1/2-Hölder continuity condition in (V.2).

Under a weak integrability condition, the unique solution to the linear SDE in (4.1) is readily recalled, by using stochastic exponentials for local martingales. For this purpose, let V be an adapted continuous process with positive paths.

(V.1) b is integrable and for any compact set K in $]0,\infty[$ there is $k_\theta \in \mathscr{L}^2(\mathbb{R}_+)$ such that $|\theta(\cdot,v)| \leq k_\theta$ for each $v \in K$ a.e.

Lemma 4.1. *Let* (V.1) *hold and χ be an $\mathscr{F}_{t_0}$-measurable random variable. Then the first SDE in* (4.1) *admits a unique solution S such that $S_{t_0} = \chi$ a.s. In fact,*

$$S_t = \chi e^{\int_{t_0}^t \theta(s,V_s)\,d\hat{W}_s + \int_{t_0}^t b(s) - \frac{1}{2}\theta(s,V_s)^2\,ds} \tag{4.2}$$

for any $t \in [t_0,T]$ a.s. In particular, if χ and $\exp(\frac{1}{2}\int_{t_0}^T \theta(s,V_s)^2\,ds)$ are integrable, then so is S and $E[S_t] = E[\chi]\exp(\int_{t_0}^t b(s)\,ds)$ for all $t \in [t_0,T]$.

In view of the preceding lemma, for any $\mathscr{F}_{t_0}$-measurable positive random variable χ, there exists a unique solution S to the first SDE in (4.1) with positive paths such that $S_{t_0} = \chi$ a.s. Then the *logarithmised process* $X := \log(S)$ satisfies

$$dX_t = \big(b(t) - (1/2)\theta(t,V_t)^2\big)\,dt + \theta(t,V_t)\,d\hat{W}_t \quad \text{for } t \in [t_0,T] \tag{4.3}$$

and it is the unique strong solution to (4.3) with $X_{t_0} = \log(\chi)$ a.s. *Growth and comparison estimates* for solutions to such SDEs with different controlling processes follow from a weak integrability and a Hölder condition on the function θ:

(V.2) There are $v_0 > 0$ and $\lambda_\theta \in \mathscr{L}^2(\mathbb{R}_+)$ such that $\theta(\cdot,v_0)$ is square-integrable and $|\theta(\cdot,v) - \theta(\cdot,\tilde{v})| \leq \lambda_\theta|v-\tilde{v}|^{1/2}$ for all $v,\tilde{v} > 0$ a.e.

This requirement implies the sublinear growth condition: $|\theta(\cdot,v)| \leq k_\theta + \lambda_\theta|v|^{1/2}$ for any $v > 0$ a.e. with $k_\theta := |\theta(\cdot,v_0)| + \lambda_\theta|v_0|^{1/2}$. Thus, if X_{t_0} and b were integrable, then the inequalities of Burkholder-Davis-Gundy, Minkowski and Young yield that

$$\begin{aligned} E\Big[\sup_{s\in[t_0,t]}|X_s|\Big] - E\big[|X_{t_0}|\big] &\leq \int_{t_0}^t |b(s)| + \frac{1}{2}E\big[\theta(s,V_s)^2\big]\,ds + 2\bigg(\int_{t_0}^t E\big[\theta(s,V_s)^2\big]\,ds\bigg)^{\frac{1}{2}} \\ &\leq c_0(t_0,t) + c_1(t_0,t)\sup_{s\in[t_0,t]} E[V_s] \end{aligned} \tag{4.4}$$

for any $t \in [t_0,T]$ with the two $\mathbb{R}_+$-valued continuous functions c_0 and c_1 defined on the set of all $(t_1,t) \in [0,T]\times[0,T]$ with $t_1 \leq t$ via

$$\begin{aligned} c_0(t_1,t) &:= \int_{t_1}^t |b(s)| + k_\theta(s)^2\,ds + 2\bigg(\int_{t_1}^t k_\theta(s)^2\,ds\bigg)^{\frac{1}{2}} + \bigg(\int_{t_1}^t \lambda_\theta(s)^2\,ds\bigg)^{\frac{1}{2}} \\ \text{and}\quad c_1(t_1,t) &:= \int_{t_1}^t \lambda_\theta(s)^2\,ds + \bigg(\int_{t_1}^t \lambda_\theta(s)^2\,ds\bigg)^{\frac{1}{2}}. \end{aligned}$$

If in addition the function $[t_0,T] \to [0,\infty]$, $t \mapsto E[V_t]$ is finite and bounded, then $\sup_{t\in[t_0,T]}|X_t|$ is integrable, entailing that X is uniformly integrable. A similar approach leads to the announced comparison bound.

Lemma 4.2. *Let* (V.2) *hold and V, $\tilde{V}$ be positive adapted continuous processes. Then any two solutions X and $\tilde{X}$ to* (4.3) *with underlying processes V and $\tilde{V}$, respectively, satisfy*

$$\begin{aligned} E\Big[\sup_{s\in[t_0,t]}|X_s^\sigma - \tilde{X}_s^\sigma|\Big] &\leq E\big[|X_{t_0} - \tilde{X}_{t_0}|\big] \\ &\quad + c_2(t_0,t)\sup_{s\in[t_0,t]}\big(1 + E[V_s^\sigma] + E[\tilde{V}_s^\sigma]\big)^{\frac{1}{2}} E\big[|V_s^\sigma - \tilde{V}_s^\sigma|\big]^{\frac{1}{2}} \end{aligned} \tag{4.5}$$

for all $t \in [t_0,T]$ and each stopping time σ with $\sigma \geq t_0$, where the $\mathbb{R}_+$-valued continuous function c_2 on the set of all $(t_1,t) \in [0,T]\times[0,T]$ with $t_1 \leq t$ is given by

$$c_2(t_1,t) := \int_{t_1}^t (k_\theta(s) + \lambda_\theta(s))\lambda_\theta(s)\,ds + 2\bigg(\int_{t_1}^t \lambda_\theta(s)^2\,ds\bigg)^{\frac{1}{2}}.$$

Let us now settle the question of pathwise uniqueness for the second SDE in (4.1) by referring to a comparison estimate, under a one-sided Lipschitz continuity condition on the drift ζ and an Osgood continuity condition on compact sets on the diffusion η:

(V.3) There is $\lambda_\zeta \in \mathscr{L}^1(\mathbb{R})$ with $\operatorname{sgn}(v-\tilde{v})(\zeta(\cdot,v)-\zeta(\cdot,\tilde{v})) \leq \lambda_\zeta|v-\tilde{v}|$ for all $v,\tilde{v} \in \mathbb{R}$ a.e.

(V.4) For each $n \in \mathbb{N}$ there are $\lambda_{\eta,n} \in \mathscr{L}^2(\mathbb{R}_+)$ and an increasing continuous function $\rho_n : \mathbb{R}_+ \to \mathbb{R}_+$ that is positive on $]0,\infty[$ such that

$$|\eta(\cdot,v) - \eta(\cdot,\tilde{v})| \leq \lambda_{\eta,n}\rho_n(|v-\tilde{v}|)$$

for any $v,\tilde{v} \in [-n,n]$ a.e. and $\int_0^1 \rho_n(v)^{-2}\,dv = \infty$.

Remark 4.3. The bound in (V.3) is valid if and only if $(\zeta(\cdot,v)-\zeta(\cdot,\tilde{v}))/(v-\tilde{v}) \leq \lambda_\zeta$ for all $v,\tilde{v} \in \mathbb{R}$ with $v \neq \tilde{v}$ a.e. For instance, this holds if $\zeta(s,\cdot)$ is locally absolutely continuous and its weak derivative $\partial_v\zeta(s,\cdot)$ is bounded from above by $\lambda_\zeta(s)$ for a.e. $s \in [0,T]$.

Under (V.3) and (V.4), an application of Corollary 3.9, combined with Remark 3.10, and Proposition 3.13 in [26] shows that there is pathwise uniqueness for the second SDE in (4.1) and any two solutions V and $\tilde{V}$ satisfy

$$E[|V_t - \tilde{V}_t|] \leq e^{\int_{t_0}^t \lambda_\zeta(s)\,ds} E[|V_{t_0} - \tilde{V}_{t_0}|] \tag{4.6}$$

for each $t \in [t_0,T]$. Regarding strong existence, let us additionally require two conditions involving the growth and continuity of the drift and diffusion coefficients ζ and η:

(V.5) There are $k_\zeta, l_\zeta \in \mathscr{L}^1(\mathbb{R})$ such that $k_\zeta \geq 0$ and $\operatorname{sgn}(v)\zeta(\cdot,v) \leq k_\zeta + l_\zeta|v|$ for any $v \in \mathbb{R}$ a.e. and $\eta(\cdot,0) = 0$.

(V.6) $\zeta(t,\cdot)$ and $\eta(t,\cdot)$ are continuous for a.e. $t \in [0,T]$. Further, for any $n \in \mathbb{N}$ there is $c_{\zeta,\eta,n} \in \mathbb{R}_+$ such that $|\zeta(\cdot,v)| \vee |\eta(\cdot,v)| \leq c_{\zeta,\eta,n}$ for each $v \in [-n,n]$ a.e.

Then Theorem 3.27 in [26] asserts that for any $\mathscr{F}_{t_0}$-measurable integrable random variable ξ the second SDE in (4.1) admits a unique strong solution V such that $V_{t_0} = \xi$ a.s. and

$$E[|V_t|] \leq e^{\int_{t_0}^t l_\zeta(s)\,ds} E[|\xi|] + \int_{t_0}^t e^{\int_s^t l_\zeta(\tilde{s})\,d\tilde{s}} k_\zeta(s)\,ds \tag{4.7}$$

for all $t \in [t_0,T]$. Now we give sufficient conditions for any solution to have a.s. positive paths. To this end, we generalise Theorem 2.2 in Mishura and Posashkova [29]. There, it is in particular required that

$$\inf_{(t,x)\in[t_0,T]\times[\delta,\infty[} \eta(t,x) > 0 \quad \text{for any } \delta > 0.$$

A positivity condition on the diffusion coefficient η, which we omit. Instead, the weakened regularity condition that we impose reads as follows:

(V.7) There are $\varepsilon > 0$ and $c_0, c_\zeta \in \mathscr{L}^1(\mathbb{R}_+)$ as well as increasing functions $\varphi_0 :]0,\varepsilon] \to \mathbb{R}_+$ and $\varphi_\zeta : [\varepsilon,\infty[\to]0,\infty[$ such that φ_ζ is continuous and

$$\frac{\eta(\cdot,v)^2}{2v^2} \leq \frac{\zeta(\cdot,v)}{v} + c_0\varphi_0(v) \quad \text{and} \quad \zeta(\cdot,\tilde{v}) \geq -c_\zeta\varphi_\zeta(\tilde{v}) \tag{4.8}$$

for each $v \in]0,\varepsilon[$ and any $\tilde{v} \geq \varepsilon$ a.e.

Regardless of whether uniqueness in law holds for the underlying equation, under this condition any solution starting at a positive deterministic value remains positive.

Proposition 4.4. *Let* (V.7) *be valid. Then any solution V to the second SDE in* (4.1) *such that $V_{t_0} = v_0$ a.s. for some $v_0 > 0$ satisfies $V_t > 0$ for any $t \in [t_0, T]$ a.s.*

Example 4.5. For $n \in \mathbb{N}$ let $k, l_1, \dots, l_n \in \mathscr{L}^1(\mathbb{R})$ and $\varphi_1, \dots, \varphi_n$ be real-valued Borel measurable functions on $]0, \infty[$ such that $k \geq 0$,

$$\zeta(t, v) = k(t) + l_1(t)\varphi_1(v) + \cdots + l_n(t)\varphi_n(v) \quad \text{and} \quad \limsup_{v \downarrow 0} \frac{|\varphi_i(v)|}{v} < \infty$$

for all $t \in [0, T]$, any $v > 0$ and each $i \in \{1, \dots, n\}$. Suppose that there are $\varepsilon_0 > 0$, $c_\eta \in \mathscr{L}^2(\mathbb{R}_+)$, $\gamma \in [1/2, \infty[$ and an increasing function $\varphi :]0, \infty[\to \mathbb{R}_+$ satisfying

$$|\eta(t, v)| \leq c_\eta(t) v^\gamma (1 + v\varphi(v))^{\frac{1}{2}} \quad \text{for any } (t, v) \in [0, T] \times]0, \varepsilon_0[. \tag{4.9}$$

If $c_\eta^2/2 \leq k$ for $\gamma = 1/2$ and, less restrictively, $c_\eta^2 \delta \leq k$ for some $\delta > 0$ whenever $\gamma \in]1/2, 1[$, then the first inequality in (4.8) holds. Indeed, take $\varepsilon \in]0, \varepsilon_0]$ and $c > 0$ such that

$$|\varphi_i(v)| \leq cv, \quad \text{and} \quad v^{2\gamma - 1} \leq 2\delta \quad \text{in case } \gamma \in]1, 2, 1[,$$

for any $i \in \{1, \dots, n\}$ and all $v \in]0, \varepsilon_0]$. Then $c_0 := c \sum_{l=1}^n |l_i| + c_\eta^2/2$ and $\varphi_0 :]0, \varepsilon] \to \mathbb{R}_+$ given by $\varphi_0(v) := 1 + v^{2\gamma - 2}(\mathbb{1}_{[1, \infty[}(\gamma) + v\varphi(v))$ satisfy

$$\frac{c_\eta(t)^2}{2v} v^{2\gamma - 1}(1 + v\varphi(v)) \leq \frac{\zeta(t, v)}{v} + c_0(t)\varphi_0(v)$$

for all $(t, v) \in [0, T] \times]0, \varepsilon[$. In particular, we may take $\alpha \in [1, \infty[^n$, $\beta \in [1/2, \infty[^n$ and $\lambda_1, \dots, \lambda_n \in \mathscr{L}^2(\mathbb{R})$ such that $\varphi_1(v) = v^{\alpha_1}, \dots, \varphi_n(v) = v^{\alpha_n}$ and

$$\eta(\cdot, v) = \lambda_1 v^{\beta_1} + \cdots + \lambda_n v^{\beta_n} \quad \text{for all } v > 0. \tag{4.10}$$

In this case, the estimate (4.9) holds for the choice $c_\eta = \sum_{i=1}^n |\lambda_i|$, $\gamma = \min_{i \in \{1, \dots, n\}} \beta_i$ and $\varphi = 0$ as soon as $\varepsilon_0 < 1$.

Due to the integrability condition $\int_0^T (1 - \rho(s)^2)^{-1} \, ds < \infty$, there is another standard $(\mathscr{F}_t)_{t \in [0, T]}$-Brownian motion W that is independent of $\tilde{W}$ such that

$$\hat{W}_t = \int_0^t \sqrt{1 - \rho(s)^2} \, dW_s + \int_0^t \rho(s) \, d\tilde{W}_s \quad \text{for all } t \in [0, T] \text{ a.s.}$$

By using this representation, a simple transformation shows that we can rearrange (4.1) into the two-dimensional SDE

$$d\begin{pmatrix} X_t \\ V_t \end{pmatrix} = \begin{pmatrix} b(t) - \frac{1}{2}\theta(t, V_t)^2 \\ \zeta(t, V_t) \end{pmatrix} dt + \begin{pmatrix} \theta(t, V_t)\sqrt{1 - \rho(t)^2} & \theta(t, V_t)\rho(t) \\ 0 & \eta(t, V_t) \end{pmatrix} d\begin{pmatrix} W_t \\ \tilde{W}_t \end{pmatrix} \tag{4.11}$$

for $t \in [t_0, T]$. Then the pair of two adapted continuous processes X and V is a solution to this SDE if and only if $(\exp(X), V)$ solves the initial one.

Further, (4.11) induces a linear differential operator $\mathscr{L}_{b,\zeta}$ on $C^{1,2}([0, T[\times \mathbb{R} \times]0, \infty[)$ with values in the linear space of all real-valued measurable functions by

$$\begin{aligned} \mathscr{L}_{b,\zeta}(\varphi)(t, x, v) &:= \Big(b(t) - \frac{1}{2}\theta(t, v)^2\Big)\frac{\partial \varphi}{\partial x}(t, x, v) + \zeta(t, v)\frac{\partial \varphi}{\partial v}(t, x, v) \\ &+ \frac{1}{2}\theta(t, v)^2 \frac{\partial^2 \varphi}{\partial x^2}(t, x, v) + \theta(t, v)\eta(t, v)\rho(t)\frac{\partial^2 \varphi}{\partial x \partial v}(t, x, v) + \frac{1}{2}\eta(t, v)^2 \frac{\partial^2 \varphi}{\partial v^2}(t, x, v). \end{aligned} \tag{4.12}$$

This formula is obtained by multiplying the diffusion coefficient with its transpose, as we readily recall, and for every solution (X,V) to (4.11) Itô's formula entails that the process $[t_0,T]\times\Omega\to\mathbb{R}$,

$$(t,\omega)\mapsto\varphi(t,X_t,V_t)(\omega)-\int_{t_0}^t\left(\frac{\partial}{\partial s}+\mathscr{L}_{b,\zeta}\right)(\varphi)(s,X_s,V_s)(\omega)\,ds$$

is a martingale for any $\varphi\in C_0^{1,2,2}([0,T]\times\mathbb{R}\times]0,\infty[)$. We will now show that (4.11) yields a time-inhomogeneous Markov process with the right-hand Feller property in the sense of [25][Section 2.3], which will allow us to apply the results on mild solutions therein.

In fact, what we get is a continuous strong Markov process, or in short, a *diffusion*, and we will realise it on the canonical space $\hat{\Omega}$ of all $\mathbb{R}\times]0,\infty[$-valued continuous paths on $[0,T]$, endowed with its Borel σ-field $\hat{\mathscr{F}}$. Let $\hat{X}:[0,T]\times\hat{\Omega}\to\mathbb{R}$ and $\hat{V}:[0,T]\times\hat{\Omega}\to]0,\infty[$ be given by

$$\hat{X}_t(\hat{\omega}):=\hat{\omega}_1(t)\quad\text{and}\quad\hat{V}_t(\hat{\omega}):=\hat{\omega}_2(t).$$

Then $(\hat{X},\hat{V})$ serves as canonical process, its natural filtration $(\hat{\mathscr{F}}_t)_{t\in[0,T]}$ satisfies $\hat{\mathscr{F}}=\hat{\mathscr{F}}_T$ and the law of any two processes $X:[0,T]\times\Omega\to\mathbb{R}$ and $V:[0,T]\times\Omega\to]0,\infty[$ is of the form $P((X,V)\in\hat{B})=P\circ(X,V)^{-1}((\hat{X},\hat{V})\in\hat{B})$ for all $\hat{B}\in\hat{\mathscr{F}}$.

Proposition 4.6. *Under* (V.1)-(V.7)*, the following four assertions hold:*

(i) We have pathwise uniqueness for the SDE (4.11)*.*

(ii) For any $(x_0,v_0)\in\mathbb{R}\times]0,\infty[$ there is a unique strong solution $(X^{t_0,x_0,v_0},V^{t_0,v_0})$ to (4.11) *such that $(X_{t_0}^{t_0,x_0,v_0},V_{t_0}^{t_0,v_0})=(x_0,v_0)$ a.s. and $V^{t_0,v_0}>0$.*

(iii) The map $[0,t]\times\mathbb{R}\times]0,\infty[\to\mathbb{R}$, $(s,x,v)\mapsto E[\varphi(s,X_t^{s,x,v},V_t^{s,v})]$ is right-continuous for any $t\in]0,T]$ and each $\varphi\in C_b([0,T]\times\mathbb{R}\times]0,\infty[)$.

(iv) Let $P_{s,x,v}$ be the law of the process $[0,T]\times\Omega\to\mathbb{R}\times]0,\infty[$, $(t,\omega)\mapsto(X_{t\vee s}^{s,x,v},V_{t\vee s}^{s,v})(\omega)$ for any $(s,x,v)\in[0,T]\times\mathbb{R}\times]0,\infty[$ and denote the set of all these measures by $\mathbb{P}$. Then $((\hat{X},\hat{V}),(\hat{\mathscr{F}}_t)_{t\in[0,T]},\mathbb{P})$ is a diffusion that is right-hand Feller.

Example 4.7. Let $\hat{\theta}_0,\hat{\theta}\in\mathscr{L}^2(\mathbb{R})$, $n\in\mathbb{N}$, $\alpha\in[1,\infty[^n$ and $\beta\in[1/2,\infty[^n$. Assume that the functions $k,l_0:[0,T]\to\mathbb{R}$ and the maps $l,\lambda:[0,T]\to\mathbb{R}^n$ are measurable and bounded such that $\theta(\cdot,v)=\hat{\theta}_0+\hat{\theta}\sqrt{v}$,

$$\zeta(\cdot,v)=k-l_0v+l_1v^{\alpha_1}+\cdots+l_nv^{\alpha_n}\quad\text{and}\quad\eta(\cdot,v)=\lambda_1v^{\beta_1}+\cdots+\lambda_nv^{\beta_n}$$

for any $v>0$. Further, let $k\geq 0$ and $l_i\leq 0$ for all $i\in\{1,\dots,n\}$ and we require that for $\gamma:=\min\{\beta_1,\dots,\beta_n\}$ it holds that $(\sum_{i=1}^n|\lambda_i|)^2/2\leq k$, if $\gamma=1/2$, and

$$\left(\sum_{i=1}^n|\lambda_i|\right)^2\delta\leq k\quad\text{for some }\delta>0\text{, if }\gamma\in]1/2,1[.$$

By imposing a radial representation of θ and η on $[0,T]\times\mathbb{R}$ and setting $\zeta(\cdot,v)=\zeta(\cdot,0)$ for all $v<0$, the SDE (4.1) reduces to

$$\begin{aligned}dS_t&=b(t)S_t\,dt+(\hat{\theta}_0(t)+\hat{\theta}(t)|V_t|^{\frac{1}{2}})S_t\,d\hat{W}_t,\\ dV_t&=\left(k(t)-l_0(t)V_t+\sum_{i=1}^n l_i(t)(V_t^+)^{\alpha_i}\right)dt+\sum_{i=1}^n\lambda_i(t)|V_t|^{\beta_i}\,d\tilde{W}_t\end{aligned}\tag{4.13}$$

for $t\in[t_0,T]$ with initial conditions $S_{t_0}=s_0$ and $V_{t_0}=v_0$ a.s. for some $s_0,v_0>0$. In particular, for $n=1$ and $l=0$ we recover the dynamics in a generalised time-dependent version of the following option pricing models:

(1) The *stock price model* by Black and Scholes [8] for $\hat{\theta}_0 = k = l_0 = \lambda = 0$ and $\hat{\theta} = 1$, which entails that any solution V to the second SDE in (4.13) satisfies $V = v_0$ a.s., where $\sqrt{v_0}$ stands for the *volatility*.

(2) The *stochastic volatility model by Heston* [23] for $\hat{\theta}_0 = 0$, $l_0 > 0$ and $\beta = 1/2$, in which case V is a *square-root diffusion*. There, l_0 is the *mean reversion speed*, k/l_0 is the *mean reversion level* and the same positivity condition $\lambda^2 \leq 2k$ for V applies.

(3) The *Garch diffusion model* [28] for $\hat{\theta}_0 = 0$, $l_0 > 0$ and $\beta = 1$. Similarly to the Heston model, l_0 is the mean reversion speed and k/l_0 the mean reversion level.

We observe that (V.1)-(V.7) hold. In fact, because the function $\mathbb{R} \to \mathbb{R}_+$, $v \mapsto (v^+)^\gamma$ is increasing, non-positive on $]-\infty, 0[$ and non-negative on $]0,\infty[$ for any $\gamma > 0$, the one-sided conditions (V.3) and (V.5) follow. More precisely,

$$\mathrm{sgn}(v - \tilde{v})(\zeta(\cdot, v) - \zeta(\cdot, \tilde{v})) \leq -l_0|v - \tilde{v}| \quad \text{and} \quad \mathrm{sgn}(v)\zeta(\cdot, v) \leq k - l_0 v$$

for every $v, \tilde{v} > 0$ a.s. From Example 4.5 we infer the validity of (V.7) and the other conditions are readily checked. Consequently, Proposition 4.6 applies to the SDE (4.13).

4.2 The logarithmised pre-default valuation PDE

Now we combine the stochastic volatility model of the previous section with the market model from Section 3 to *characterise pre-default value processes via mild solutions* to the associated parabolic PDE.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \tilde{P})$ serve as underlying probability space such that the usual conditions hold. We assume that both S and V take positive values only and consider the following specifications for our market model:

(P.1) $G(\tau_{\mathcal{I}})$ and $G(\tau_{\mathcal{C}})$ are deterministic, absolutely continuous, positive and satisfy (3.5), $\tau_{\mathcal{I}}$ and $\tau_{\mathcal{C}}$ are $(\mathscr{F}_t)_{t\in[0,T]}$-conditionally independent under $\tilde{P}$ and $G(\tau) = G(\tau_{\mathcal{I}})G(\tau_{\mathcal{C}})$.

(P.2) The risk-free rate r, the remuneration rates ${}_+c$, ${}_-c$, the funding rates ${}_+\tilde{f}$, ${}_-\tilde{f}$ and the hedging rates ${}_+\tilde{h}$, ${}_-\tilde{h}$, which have integrable paths, are deterministic. That is,

$$r_t = \hat{r}(t), \quad {}_ic_t = \hat{c}_i(t), \quad {}_i\tilde{f}_t = \hat{f}_i(t), \quad {}_i\tilde{h}_t = \hat{h}_i(t)$$

for every $t \in [0,T]$, both $i \in \{+,-\}$ and some $\hat{r}, \hat{c}_i, \hat{f}_i, \hat{h}_i \in \mathscr{L}^1(\mathbb{R})$.

(P.3) There is a measurable function $\phi :]0,\infty[^2 \to \mathbb{R}_+$ and a real-valued continuous function $\hat{\pi}$ on $[0,T]\times]0,\infty[^2$ such that $\Phi(s,v) = \phi(s(T), v(T))$ for any $s, v \in C([0,T])$ that are positive and $\pi = \hat{\pi}(\cdot, S, V)$.

(P.4) There are $\alpha, \beta \in C([0,T])$ with $0 \leq \alpha \leq \beta \leq 1$ and some continuous function $\hat{H} : [0,T]\times]0,\infty[^2\times\mathbb{R} \to \mathbb{R}$ satisfying $C(Y) = \alpha Y$, $\varepsilon(Y) = \beta Y$,

$$\tilde{H}(Y) = \hat{H}(\cdot, S, V, Y) \quad \text{and} \quad \tilde{F}(Y) = (1-\alpha)Y \quad \text{for all } Y \in \mathscr{S}.$$

Remark 4.8. Under (P.1) and (P.2), Remark 3.3 entails that any $\mathscr{F}_T$-measurable random variable lies in $\tilde{\mathscr{L}}(r,\tau)$ if and only if it is $\tilde{P}$-integrable. Further, if Y is a process that is $(\mathscr{F}_t)_{t\in[0,T]}$-progressively measurable, then $Y \in \tilde{\mathscr{S}}(r,\tau) \Leftrightarrow \int_0^T \tilde{E}[|Y_t|]\,dt < \infty$.

It is readily seen that (P.1) holds in Example 3.9 as soon as the two processes $\lambda^{(\mathcal{I})}$ and $\lambda^{(\mathcal{C})}$ there are deterministic. Put differently, for both $i \in \{\mathcal{I}, \mathcal{C}\}$ there is $\hat{\lambda}_i \in \mathscr{L}^1(\mathbb{R}_+)$ such that $\lambda^{(i)} = \hat{\lambda}_i$ and $\int_0^t \hat{\lambda}_i(s)\,ds > 0$ for some $t \in]0,T]$.

Further, (P.1) and (P.2) imply (M.2) and (M.4), respectively. If, conversely, (M.4) is satisfied and the pre-default rates ${}_{+}f$, ${}_{-}f$, ${}_{+}h$, ${}_{-}h$ are deterministic, then, as required in (P.2), so are the rates ${}_{+}\tilde{f}$, ${}_{-}\tilde{f}$, ${}_{+}\tilde{h}$, ${}_{-}\tilde{h}$, according to (3.4).

In (P.3) the payoff functional Φ is reduced to a function of the terminal state and the dividend rate π to a function of the current state of the process $[0,T]\times\Omega \to [0,T]\times]0,\infty[^2$, $(t,\omega) \mapsto (t, S_t, V_t)(\omega)$. So, if (P.1) and (P.2) are valid, then (M.1) holds if and only if

$$\tilde{E}[\phi(S_T, V_T)] \quad \text{and} \quad \int_0^T \tilde{E}[|\hat{\pi}(t, S_t, V_t)|]\, dt \quad \text{are finite,} \tag{4.14}$$

as Remark 4.8 shows. Now let for the moment (P.4) be valid. Then we can use $\hat{H}(\cdot, S, V, Y)$ as pre-default version of $\hat{H}(\cdot, S, V, \tilde{Y})$ for any $\tilde{Y} \in \tilde{\mathscr{S}}$ that is integrable up to time τ with pre-default version Y if

$$\tilde{E}[|\hat{H}(t, S_t, V_t, Y_t)|] < \infty \quad \text{for all } t \in [0,T]. \tag{4.15}$$

Under this condition, (M.3) follows from (P.4). Hence, let us now assume that (P.1)-(P.4), (4.14) and (4.15) hold. Then (M.1)-(M.4) are satisfied and we may turn to the pre-default valuation in (VE).

Moreover, (P.4) now ensures that the collateral process $C(\mathscr{V})$ and the close-out value $\varepsilon(\mathscr{V})$ are deterministic ordered fractions of any given pre-default value process $\mathscr{V}$ and the financing hypothesis (3.26) holds.

By recalling the time-dependent random functionals ${}_{0}\mathrm{B}$, ${}_{\mathcal{I}}\mathrm{B}$ and ${}_{\mathcal{C}}\mathrm{B}$ in (3.16), we see that the random functional B given by (3.24) is of the form

$$\begin{aligned}
\mathrm{B}_t(Y) &= \pi_t - (c_t(Y) - r_t)C_t(Y) - (f_t(Y) - r_t)F_t(Y) - (r_t - h_t(Y))H_t(Y) - r_t Y_t \\
&\quad - \frac{\dot{G}_t(\tau_{\mathcal{I}})}{G_t(\tau_{\mathcal{I}})}\big(\varepsilon_t(Y) + \mathrm{LGD}_{\mathcal{I}}((\varepsilon_t - C_t)^- + F_t^+)(Y)\mathbb{1}_{\{\mathcal{I}=\mathcal{B}\}} - Y_t\big) \\
&\quad - \frac{\dot{G}_t(\tau_{\mathcal{C}})}{G_t(\tau_{\mathcal{C}})}\big(\varepsilon_t(Y) - \mathrm{LGD}_{\mathcal{C}}(\varepsilon_t - C_t)^+(Y) - Y_t\big) = \hat{B}(t, S_t, V_t, Y_t)
\end{aligned}$$

for all $t \in [0,T]$ and each continuous $Y \in \mathscr{S}$ with the real-valued measurable function $\hat{B}$ defined on $[0,T]\times]0,\infty[^2\times\mathbb{R}$ via

$$\begin{aligned}
\hat{B}(t,s,v,y) &:= \hat{\pi}(t,s,v) - (\hat{c}_+\alpha + \hat{f}_+(1-\alpha))(t)y^+ + (\hat{c}_-\alpha + \hat{f}_-(1-\alpha))(t)y^- \\
&\quad - (\hat{r} - \hat{h}_+)(t)\hat{H}^+(t,s,v,y) + (\hat{r} - \hat{h}_-)(t)\hat{H}^-(t,s,v,y) \\
&\quad + g_{\mathcal{I}}(t)\Big((1-\beta)(t)y - \mathrm{LGD}_{\mathcal{I}}((\beta-\alpha)(t)y^- + (1-\alpha)(t)y^+)\mathbb{1}_{\{\mathcal{B}\}}(\mathcal{I})\Big) \\
&\quad + g_{\mathcal{C}}(t)\Big((1-\beta)(t)y + \mathrm{LGD}_{\mathcal{C}}(\beta-\alpha)(t)y^+\Big),
\end{aligned} \tag{4.16}$$

where $g_i : [0,T] \to \mathbb{R}$ is a measurable integrable function satisfying $g_i = \dot{G}(\tau_i)/G(\tau_i)$ for both $i \in \{\mathcal{I}, \mathcal{C}\}$. In this deterministic setting, all the cash flows and costs and benefits

$${}_{\mathrm{con}}\mathrm{CF}(\mathscr{V}), \quad {}_{\mathrm{col}}\mathrm{C}(\mathscr{V}), \quad {}_{\mathrm{fun}}\mathrm{C}(\mathscr{V}), \quad {}_{\mathrm{hed}}\mathrm{C}(\mathscr{V}) \quad \text{and} \quad {}_{\mathrm{def}}\mathrm{CF}(\mathscr{V}, \tilde{\mathscr{V}})$$

are measurable functionals of $(\tau_{\mathcal{I}}, \tau_{\mathcal{C}}, S, V, \mathscr{V}, \tilde{\mathscr{V}})$ for any $\tilde{\mathscr{V}} \in \tilde{\mathscr{S}}$ that is integrable up to time τ with pre-default version $\mathscr{V}$. Further, the continuous process $A(\mathscr{V})$, given by (3.17) and appearing in (VE), is adapted to the natural filtration of $(S, V, \mathscr{V})$ and we may now deal with the assumed existence of $\tilde{P}$.

So, suppose that (4.1) is solved strongly by S and V on $[t_0, T]$ under P for some $t_0 \in [0,T]$ and (V.1) holds. Let λ, $\tilde{\lambda}$ be two $(\mathscr{F}_t)_{t\in[t_0,T]}$-progressively measurable processes with square-integrable paths and define a continuous local martingale Z via

$$Z^{t_0} = 1 \quad \text{and} \quad Z_t = \exp\Big(-\int_{t_0}^t \lambda_s\, dW_s - \int_{t_0}^t \tilde{\lambda}_s\, d\tilde{W}_s - \frac{1}{2}\int_{t_0}^t \lambda_s^2 + \tilde{\lambda}_s^2\, ds\Big) \tag{4.17}$$

for any $t \in [t_0,T]$ a.s. Under the condition that $E[Z_T]=1$, Girsanov's theorem states that $W^{(\lambda)} := W + \int_{t_0}^{\cdot\vee t_0}\lambda_s\,ds$ and $W^{(\tilde{\lambda})} := \tilde{W} + \int_{t_0}^{\cdot\vee t_0}\tilde{\lambda}_s\,ds$ are independent $(\mathscr{F}_t)_{t\in[0,T]}$-Brownian motions under the measure $\hat{P}_{t_0,\lambda,\tilde{\lambda}}$ on $(\Omega,\mathscr{F})$ given by $\hat{P}_{t_0,\lambda,\tilde{\lambda}}(A) := E[Z_T\mathbb{1}_A]$.

Assuming that S is the only price process to consider, Remark 3.2 and (P.1) entail that $\hat{P}_{t_0,\lambda,\tilde{\lambda}}$ is an equivalent local martingale measure after time t_0 if and only if

$$(b-\hat{r})(t) = \theta(t,V_t)\big(\lambda_t\sqrt{1-\rho(t)^2} + \tilde{\lambda}_t\rho(t)\big) \quad \text{for a.e. } t\in[t_0,T] \text{ a.s.}$$

Indeed, this follows from the representation (4.2) in Lemma 4.1 and the relation between $\hat{W}$ and $(W,\tilde{W})$, stated directly before (4.11). Thus, if $\theta(\cdot,v)>0$ for all $v>0$, then we propose to take the *market prices of risk*

$$\tilde{\lambda}_t = \gamma\theta(t,V_t) \quad \text{and} \quad \lambda_t = \left(\frac{(b-\hat{r})(t)}{\theta(t,V_t)} - \gamma\theta(t,V_t)\rho(t)\right)\frac{1}{\sqrt{1-\rho(t)^2}} \tag{4.18}$$

for all $t\in[t_0,T]$ and some $\gamma\in\mathbb{R}_+$. As V is a strong solution, λ is independent of W. Hence, if b, $\hat{r}$, ρ and θ are continuous, then it follows from Theorem 21.4 in [1] and Lemma 35 in [16] that

$$E[Z_T] = E\bigg[\exp\Big(-\gamma\int_{t_0}^T \theta(t,V_t)\,d\tilde{W}_t - \frac{1}{2}\gamma^2\int_{t_0}^T\theta(t,V_t)^2\,dt\Big)\bigg],$$

and for $E[Z_T]=1$ to hold, it suffices that $\exp(\frac{\gamma^2}{2}\int_{t_0}^T\theta(t,V_t)^2\,dt)$ is P-integrable, by Novikov's condition. In this case, we set $\tilde{P}_{t_0,V,\gamma} := \hat{P}_{t_0,\lambda,\tilde{\lambda}}$ and for the log-price process $X=\log(S)$ we see that (X,V) solves the SDE

$$d\begin{pmatrix}X_t\\ V_t\end{pmatrix} = \begin{pmatrix}\hat{r}(t)-\frac{1}{2}\theta(t,V_t)^2\\ (\zeta-\gamma\hat{\eta}\hat{\theta})(t,V_t)\end{pmatrix}dt + \begin{pmatrix}\theta(t,V_t)\sqrt{1-\rho(t)^2} & \theta(t,V_t)\rho(t)\\ 0 & \eta(t,V_t)\end{pmatrix} d\begin{pmatrix}W_t^{(\lambda)}\\ W_t^{(\tilde{\lambda})}\end{pmatrix} \tag{4.19}$$

for $t\in[t_0,T]$ under $\tilde{P}_{t_0,V,\gamma}$, where $\hat{\eta},\hat{\theta}:[0,T]\times\mathbb{R}\to\mathbb{R}$ are given by $\hat{\eta}(t,v):=\eta(t,v^+)$ and $\hat{\theta}(t,v):=\theta(t,v^+)$, since the values of η and θ on $[0,T]\times]-\infty,0[$ are irrelevant. In particular, as $\gamma=0$ feasible, there exists an equivalent local martingale measure, and we refer to Section 3 in [35] for a more specific analysis in the Heston model.

Under certain conditions, we now show that if $\mathscr{V}\in\mathscr{S}$ is of the form $\mathscr{V}_t = u(t,X_t,V_t)$ for all $t\in[0,T]$ and some continuous function $u:[0,T]\times\mathbb{R}\times]0,\infty[\to\mathbb{R}$, then it solves (VE) as soon as u is a mild solution to the parabolic semilinear PDE

$$\frac{\partial u}{\partial t}(t,x,v) + \mathscr{L}_{\hat{r},\zeta-\gamma\hat{\eta}\hat{\theta}}(u)(t,x,v) = -\hat{B}(t,e^x,v,u(t,x,v)) \tag{4.20}$$

for all $(t,x,v)\in[0,T[\times\mathbb{R}\times]0,\infty[$ with terminal value condition $u(T,\cdot,\cdot)=\phi(\exp(\cdot),\cdot)$. Thereby, $\mathscr{L}_{\hat{r},\zeta-\gamma\hat{\eta}\hat{\theta}}$ stands for the differential operator (4.12) when (b,ζ) is replaced by $(\hat{r},\zeta-\gamma\hat{\eta}\hat{\theta})$ with the arbitrarily chosen $\gamma\in\mathbb{R}_+$.

For introduction of the mild solution concept and the resulting characterisation of value processes, we first ensure that Proposition 4.6 applies to any two-dimensional SDE whose coefficients agree with those of (4.19), by introducing the following condition:

(V.8) If $\gamma>0$, then θ is continuous, $\theta(\cdot,0)=0$ and there are $k_{\theta,\eta},l_{\theta,\eta},\lambda_{\theta,\eta}\in\mathscr{L}^1(\mathbb{R})$, $\varepsilon_0>0$ and $\lambda_{\eta,0}\in\mathscr{L}^2(\mathbb{R}_+)$ such that

$$(\eta\theta)(\cdot,v)\geq k_{\theta,\eta}+l_{\theta,\eta}v \quad \text{and} \quad \mathrm{sgn}(v-\tilde{v})\big((\eta\theta)(\cdot,v)-(\eta\theta)(\cdot,\tilde{v})\big)\geq\lambda_{\theta,\eta}|v-\tilde{v}| \tag{4.21}$$

for any $v,\tilde{v}\in\mathbb{R}_+$ and $|\eta(\cdot,v)|\leq\lambda_{\eta,0}v^{1/2}$ for all $v\in]0,\varepsilon_0[$ a.e.

Remark 4.9. Let $\theta(t,\cdot)$ and $\eta(t,\cdot)$ be non-negative and increasing on $\mathbb{R}_+$ for a.e. $t \in [0,T]$, $\eta(\cdot,0) = 0$ and (V.4) be valid for $\rho_1(v) = v^{1/2}$ for all $v \in \mathbb{R}_+$. Then (4.21) and the succeeding bound on η hold for $k_{\theta,\eta} = l_{\theta,\eta} = \lambda_{\theta,\eta} = 0$, $\varepsilon_0 = 1$ and $\lambda_{\eta,0} = \lambda_{\eta,1}$.

Now we assume that (V.1)-(V.8) hold, in which case the same conditions (V.1)-(V.7) follow for $\hat{r}$ and $\zeta - \gamma\hat{\theta}\hat{\eta}$ instead of b and ζ, respectively, which shows that Proposition 4.6 also covers the type of two-dimensional SDE that we just derived.

Thus, for any $(s,x,v) \in [0,T]\times\mathbb{R}\times]0,\infty[$ let $(\tilde{X}^{s,x,v}, \tilde{V}^{s,v})$ be a strong solution to (4.11) on $[s,T]$ when (b,ζ) is replaced by $(\hat{r}, \zeta - \gamma\hat{\eta}\hat{\theta})$ such that $(\tilde{X}_s^{s,x,v}, \tilde{V}_s^{s,v}) = (x,v)$ a.s. and $\tilde{V}^{s,v} > 0$. Further, denote the law of the continuous process

$$[0,T]\times\Omega \to \mathbb{R}\times]0,\infty[, \quad (t,\omega) \mapsto (\tilde{X}_{t\vee s}^{s,x,v}, \tilde{V}_{t\vee s}^{s,v})(\omega)$$

by $\tilde{P}_{s,x,v}$ and recall the canonical process $(\hat{X},\hat{V})$ from Section 4.1. Then, for a measurable function $\tilde{B} : [0,T]\times]0,\infty[^2\times\mathbb{R} \to \mathbb{R}$ a *mild solution* to the terminal value problem (4.20) with $\tilde{B}$ instead of $\hat{B}$ is a measurable function $u : [0,T]\times\mathbb{R}\times]0,\infty[\to \mathbb{R}$ for which

$$\int_s^T |\tilde{B}(t, \exp(\hat{X}_t), \hat{V}_t, u(t,\hat{X}_t,\hat{V}_t))|\,dt$$

is finite and $\tilde{P}_{s,x,v}$-integrable such that the implicit integral equation

$$\begin{aligned}\tilde{E}_{s,x,v}[\phi(\exp(\hat{X}_T),\hat{V}_T)] &= u(s,x,v)\\ &\quad - \tilde{E}_{s,x,v}\bigg[\int_s^T \tilde{B}(t,\exp(\hat{X}_t),\hat{V}_t,u(t,\hat{X}_t,\hat{V}_t))\,dt\bigg]\end{aligned} \tag{4.22}$$

holds for any $(s,x,v) \in [0,T]\times\mathbb{R}\times]0,\infty[$. Provided that ϕ is continuous, we also recall that a *classical solution* to this terminal value problem is a real-valued continuous function u on $[0,T]\times\mathbb{R}\times]0,\infty[$ that lies in $C^{1,2,2}([0,T[\times\mathbb{R}\times]0,\infty[)$ and satisfies (4.20) and $u(T,\cdot,\cdot) = \phi(\exp(\cdot),\cdot)$.

Thereby, we stress the fact that if $\tilde{B}$ satisfies the two conditions (4.25) and (4.26) considered below and ϕ is bounded and continuous, then every bounded classical solution is also a mild solution. For instance, see Section 2.4 in [27] for a concise justification.

Next, as we will use any of the derived local martingale measures, let $(X^{s,x,v}, V^{s,v})$ be a strong solution to (4.11) on $[s,T]$ satisfying $(X_s^{s,x,v}, V_s^{s,v}) = (x,v)$ a.s. and $V^{s,v} > 0$, and write $P_{s,x,v}$ for the law of the process $[0,T]\times\Omega \to \mathbb{R}\times]0,\infty[$, $(t,\omega) \mapsto (X_{t\vee s}^{s,x,v}, V_{t\vee s}^{s,v})(\omega)$ for each $(s,x,v) \in [0,T]\times\mathbb{R}\times]0,\infty[$, just as in Proposition 4.6. Let us also recall the regularity conditions that we used for this derivation:

(V.9) b, $\hat{r}$, ρ and θ are continuous and $\theta(\cdot,v) > 0$ for all $v > 0$.

Now we come to one of the main results of our work, *a characterisation of value processes in terms of mild solutions*, which motivates to call any mild solution to the terminal value problem (4.20) *a pre-default valuation function*.

Theorem 4.10. *Let* (V.1)-(V.9) *and* (P.1)-(P.4) *hold,* ϕ *be bounded and*

$$\int_s^T \tilde{E}_{s,x,v}[|\hat{\pi}(t,\exp(\hat{X}_t),\hat{V}_t)|]\,dt \quad \text{and} \quad E_{s,x,v}\bigg[\exp\bigg(\frac{\gamma^2}{2}\int_s^T \theta(t,\hat{V}_t)^2\,dt\bigg)\bigg] \tag{4.23}$$

be finite for all $(s,x,v) \in [0,T]\times\mathbb{R}\times]0,\infty[$. *Further, let* $u \in C_b([0,T]\times\mathbb{R}\times]0,\infty[)$ *and define* $\mathscr{V} \in \mathscr{S}$ *by* $\mathscr{V}_t := u(t,X_t,V_t)$. *Then the following two assertions hold:*

(i) Suppose for each $(s,x,v) \in [0,T]\times\mathbb{R}\times]0,\infty[$ *that*

$$\sup_{t\in[s,T]} \tilde{E}_{s,x,v}[|\hat{H}(t,\exp(\hat{X}_t),\hat{V}_t,u(t,\hat{X}_t,\hat{V}_t))|] < \infty, \tag{4.24}$$

and $\mathscr{V}$ *solves* (VE) *on* $[s,T]$ *whenever* (X,V) *is a solution to* (4.11) *on* $[s,T]$ *with* $(X^s,V^s) = (x,v)$ *a.s. and* $\tilde{P} = \tilde{P}_{s,V,\gamma}$. *Then* u *is a mild solution to* (4.20).

(ii) *Conversely, let u be a mild solution to* (4.20) *and* $(s,x,v) \in [0,T]\times\mathbb{R}\times]0,\infty[$ *be such that* (4.24) *holds. If* (X,V) *solves* (4.11) *on* $[s,T]$,

$$(X^s, V^s) = (x,v) \quad a.s., \quad \tilde{P} = \tilde{P}_{s,V,\gamma}$$

and $(\mathscr{F}_t)_{t\in[0,T]}$ *is the right-continuous filtration of the augmented natural filtration of* (X,V)*, then* $\mathscr{V}$ *is a solution to* (VE) *on* $[s,T]$.

Remark 4.11. The integrability conditions in (4.23) and (4.24) on $\hat{\pi}$ and $\hat{H}$ are satisfied if there are $c_{\hat{\pi}} \in \mathscr{L}^1(\mathbb{R}_+)$ and $c_{\hat{H}} \in \mathbb{R}_+$ such that

$$|\hat{\pi}(\cdot, e^x, v)| \leq c_{\hat{\pi}}(1 + x + v) \quad \text{for any } (x,v) \in \mathbb{R}\times]0,\infty[\text{ a.e.}$$

and $|\hat{H}(\cdot, e^x, v, y)| \leq c_{\hat{H}}(1 + x + v + |y|)$ for every $(x,v,y) \in \mathbb{R}\times]0,\infty[\times\mathbb{R}$, as the two growth estimates (4.4) and (4.7) show.

For an analysis of mild solutions to semilinear PDEs such as (4.20) we will now apply results from [25]. Let $\tilde{B}$ be a real-valued measurable function on $[0,T]\times]0,\infty[^2\times\mathbb{R}$ for which the following *affine growth condition* and *Lipschitz condition on compact sets* hold: There is $c_{\tilde{B}} \in \mathscr{L}^1(\mathbb{R}_+)$ such that

$$|\tilde{B}(\cdot, e^x, v, y)| \leq c_{\tilde{B}}(1 + |y|) \tag{4.25}$$

for all $v > 0$ and any $x, y \in \mathbb{R}$ a.e. Moreover, for every $n \in \mathbb{N}$ there is $\lambda_{\tilde{B},n} \in \mathscr{L}^1(\mathbb{R}_+)$ satisfying

$$|\tilde{B}(\cdot, e^x, v, y) - \tilde{B}(\cdot, e^x, v, \tilde{y})| \leq \lambda_{\tilde{B},n}|y - \tilde{y}| \tag{4.26}$$

for all $(x,v) \in \mathbb{R}\times]0,\infty[$ and any $y, \tilde{y} \in [-n,n]$ a.e. If ϕ is bounded, then Theorem 2.15 in [25] yields a unique bounded mild solution $u_{\tilde{B},\phi}$ to the terminal value problem (4.20) when $\hat{B}$ is replaced by $\tilde{B}$.

To ensure that (4.25) and (4.26) hold for $\hat{B}$, we require that the dividend function $\hat{\pi}$ obeys a suitable bound and the pre-default hedging function $\hat{H}$ is both of affine growth and locally Lipschitz continuous in $y \in \mathbb{R}$, uniformly in $(t,s,v) \in [0,T]\times]0,\infty[^2$:

(P.5) There exist some $c_{\hat{\pi}} \in \mathscr{L}^1(\mathbb{R}_+)$ and $c_{\hat{H}} \geq 0$ satisfying $|\hat{\pi}(\cdot, \exp(x), v)| \leq c_{\hat{\pi}}$ for every $(x,v) \in \mathbb{R}\times]0,\infty[$ a.e. and

$$|\hat{H}(t, e^x, v, y)| \leq c_{\hat{H}}(1 + |y|)$$

for all $(t,x,v,y) \in [0,T]\times\mathbb{R}\times]0,\infty[\times\mathbb{R}$. Further, for each $n \in \mathbb{N}$ there is $\lambda_{\hat{H},n} \geq 0$ satisfying

$$|\hat{H}(t, e^x, v, y) - \hat{H}(t, e^x, v, \tilde{y})| \leq \lambda_{\hat{H},n}|y - \tilde{y}|$$

for each $(t,x,v) \in [0,T]\times\mathbb{R}\times]0,\infty[$ and every $y, \tilde{y} \in [-n,n]$.

Then an application of Theorem 2.15 in [25] yields *the existence and uniqueness of a mild solution, including a right-continuity and value analysis*. Thereby, J denotes an interval in $\mathbb{R}$ with $\underline{d} := \inf J$ and $\overline{d} := \sup J$.

Proposition 4.12. *Let* (V.1)-(V.8) *and* (P.1)-(P.5) *be valid and* ϕ *be bounded. Then there is a unique bounded mild solution* u_ϕ *to* (4.20) *that is right-continuous if* ϕ *is continuous. Moreover, if*

$$\underline{d} > -\infty \ (\textit{resp. } \overline{d} < \infty) \textit{ implies } \hat{B}(\cdot, e^x, v, \underline{d}) \geq 0 \ (\textit{resp. } \hat{B}(\cdot, e^x, v, \overline{d}) \leq 0) \tag{4.27}$$

for all $(x,v) \in \mathbb{R}\times]0,\infty[$ *a.e., then* u_ϕ *takes all its values in* J *as soon as* ϕ *does.*

Remark 4.13. As payoff function, ϕ is modelled to be $\mathbb{R}_+$-valued. Thus, the pre-default valuation function u_ϕ is non-negative if

$$\hat{\pi}(\cdot, e^x, v) \geq (\hat{r} - \hat{h}_+)\hat{H}^+(\cdot, e^x, v, 0) - (\hat{r} - \hat{h}_-)\hat{H}^-(\cdot, e^x, v, 0) \tag{4.28}$$

for any $(x, v) \in \mathbb{R}\times]0, \infty[$ a.e. In this case, $u_\phi > 0$ follows from $\phi > 0$.

Next, from Lemma 4.2 in [25] we obtain *a sensitivity analysis*. Let $\tilde{\phi} :]0, \infty[^2 \to \mathbb{R}$ be measurable and bounded and suppose in the setting of Proposition 4.12 that u_ϕ and $u_{\tilde{B},\tilde{\phi}}$ are J-valued. This is the case if $\phi, \tilde{\phi} \in J$ and condition (4.27) holds not only for $\hat{B}$ but also for $\tilde{B}$. Then

$$\tilde{B} \geq \hat{B} \quad \text{on } [0, T]\times]0, \infty[^2 \times J \quad \text{and} \quad \tilde{\phi} \geq \phi \quad \text{entails} \quad u_{\tilde{B},\tilde{\phi}} \geq u_\phi. \tag{4.29}$$

So, u_ϕ depends on an increasing way on the dividend function $\hat{\pi}$, the remuneration rate $\hat{c}_-$, the pre-default funding rate $\hat{f}_-$ and the pre-default hedging rate $\hat{h}_+$. At the same time, *the comparison* (4.29) shows that u_ϕ depends decreasingly on $\hat{c}_+$, $\hat{f}_+$ and $\hat{h}_-$.

If (4.28) holds, then the dependence is monotonically increasing on the term $g_{\mathcal{C}}$. To draw the same conclusion for $g_{\mathcal{I}}$, the estimate $1 - \beta \geq \mathrm{LGD}_{\mathcal{I}}(1 - \alpha)$ has to be valid when the investor $\mathcal{I}$ is a bank. This, for instance, is the case whenever the fractions α and β coincide.

Finally, to give conditions under which the mild solution $u_{\tilde{B},\tilde{\phi}}$ turns into a classical one, we use Theorem 2.4 in [2]. This leads to a local Lipschitz condition on the coefficients in (4.19) and the determinant of the diffusion coefficient should not vanish:

(V.10) $\hat{r}$ and ρ are Lipschitz continuous, ζ, η and θ are locally Lipschitz continuous on $[0, T]\times]0, \infty[$ and $|\eta|$ and $|\theta|$ are positive on $[0, T]\times]0, \infty[$.

Moreover, $\tilde{B}$ is required to be of affine growth and locally Lipschitz continuous in $y \in \mathbb{R}$, uniformly in $(t, s, v) \in [0, T]\times]0, \infty[^2$. So, the functions $c_{\tilde{B}}$ and $\lambda_{\tilde{B},n}$ in (4.25) and (4.26), respectively, should be bounded for all $n \in \mathbb{N}$. According to the assumptions in [2], the function $\tilde{B}$ has to be continuously differentiable as well. However, a short investigation of the proof therein shows that local Hölder continuity suffices.

In summary, if (V.1)-(V.8) and (V.10) hold, $\tilde{B}$ satisfies these conditions and $\tilde{\phi}$ is continuous, then [2] asserts that the terminal value problem (4.20) with $(\tilde{B}, \tilde{\phi})$ instead of $(\hat{B}, \phi)$ admits a unique bounded classical solution. This function must be $u_{\tilde{B},\tilde{\phi}}$, as Theorem 2.15 in [25] yields uniqueness among bounded mild solutions. To ensure that $\hat{B}$ meets the same requirements as $\tilde{B}$, we require the following condition:

(P.6) $\hat{\pi}$ is bounded. Further, $\hat{\pi}$, the rates $\hat{r}$, $\hat{c}_+, \hat{c}_-$, $\hat{f}_+, \hat{f}_-$, $\hat{h}_+$, $\hat{h}_-$, the functions $g_{\mathcal{I}}$, $g_{\mathcal{C}}$, the fractions α, β and the hedging function $\hat{H}$ are locally Hölder continuous.

Then Theorem 2.4 in [2] combined with Theorem 2.15 in [25] yield *sufficient conditions for the unique mild solution* of Proposition 4.12 *to be classical.*

Corollary 4.14. *Let* (V.1)-(V.8)*,* (V.10) *and* (P.1)-(P.6) *hold and ϕ be bounded and continuous. Then u_ϕ is the unique bounded classical solution to* (4.20).

5 Proofs for the preliminary results and the market model

5.1 Proofs for the representations of conditional expectations

Proof of Lemma 2.1. For $\tilde{A} \in \tilde{\mathscr{F}}_s$ there is $A \in \mathscr{F}_s$ such that $\{\rho > s\} \cap A = \{\rho > s\} \cap \tilde{A}$. Thus, the properties of conditional expectation yield that

$$E[X\mathbb{1}_{\{\rho>t\}}P(\rho > s|\mathscr{F}_s)\mathbb{1}_{\tilde{A}}] = E[E[X\mathbb{1}_{\{\rho>t\}}|\mathscr{F}_s]P(\rho > s|\mathscr{F}_s)\mathbb{1}_A]$$

$$= E[E[X\mathbb{1}_{\{\rho>t\}}|\mathscr{F}_s]\mathbb{1}_{\{\rho>s\}\cap\tilde{A}}].$$

This implies the assertion. □

Proof of Corollary 2.2. If $X = \tilde{X}$ a.s. on $\{\rho > t\}$, then the $\mathscr{F}_t$-measurability of X yields that $XG_t(\rho) = E[X\mathbb{1}_{\{\rho>t\}}|\mathscr{F}_t] = E[\tilde{X}\mathbb{1}_{\{\rho>t\}}|\mathscr{F}_t]$ a.s.

Conversely, suppose that (2.3) holds. Then $\tilde{X}P(\rho > t|\tilde{\mathscr{F}}_t) = XP(\rho > t|\tilde{\mathscr{F}}_t)$ a.s. on $\{G_t(\rho) > 0\}$, by Lemma 2.1. Thus,

$$E[\tilde{X}\mathbb{1}_{\{\rho>t\}\cap\{G_t(\rho)>0\}\cap\tilde{A}}] = E[\tilde{X}P(\rho > t|\tilde{\mathscr{F}}_t)\mathbb{1}_{\{G_t(\rho)>0\}\cap\tilde{A}}] = E[X\mathbb{1}_{\{\rho>t\}\cap\{G_t(\rho)>0\}\cap\tilde{A}}]$$

for any $\tilde{A} \in \tilde{\mathscr{F}}_t$. We first choose $\tilde{A} = \{n \geq \tilde{X} > X\}$ and then $\tilde{A} = \{\tilde{X} \leq X \leq n\}$ in this identity for each $n \in \mathbb{N}$ to infer that $X = \tilde{X}$ a.s. on $\{\rho > t\}$, since

$$P(\{\rho > t\} \cap \{G_t(\rho) = 0\}) = E[G_t(\rho)\mathbb{1}_{\{G_t(\rho)=0\}}] = 0.$$

□

Proof of Lemma 2.3. Since $E[\tilde{X}_t\mathbb{1}_{\{\rho>t\}}|\mathscr{F}_t] = X_tG_t(\rho)$ a.s. for every $t \in [s, T]$, Fubini's theorem directly yields that

$$E\bigg[\int_s^{T\wedge\rho} \tilde{X}_t\,dt\,\mathbb{1}_A\bigg] = \int_s^T E[\tilde{X}_t\mathbb{1}_{\{\rho>t\}}\mathbb{1}_A]\,dt = E\bigg[\int_s^T X_tG_t(\rho)\,dt\,\mathbb{1}_A\bigg]$$

for each $A \in \mathscr{F}_s$. Thus, the claim holds. □

Proof of Lemma 2.5. The system $\mathscr{E}$ of all sets $]s, \tilde{s}] \times A$, where $s, \tilde{s} \in [0, t]$ satisfy $s \leq \tilde{s}$ and $A \in \{\emptyset, \{\infty\}\}$, is an $\cap$-stable generator of $\mathscr{B}([0, t] \cup \{\infty\})$ and we readily check that

$$P(s_1 < \sigma \leq t_1, s_2 < \tau \leq t_2|\mathscr{F}_t) = P(s_1 < \sigma \leq t_1|\mathscr{F}_t)P(s_2 < \tau \leq t_2|\mathscr{F}_t) \quad \text{a.s.}$$

for any $s_1, t_1, s_2, t_2 \in [0, t] \cup \{\infty\}$ with $s_1 \leq t_1$ and $s_2 \leq t_2$. In particular, the d-system of all $C \in \mathscr{B}(([0, t] \cup \{\infty\})^2)$ for which (2.4) holds includes $\mathscr{E} \times \mathscr{E}$. Hence, the claim follows from the monotone class theorem. □

Proof of Proposition 2.6. For fixed $\tilde{s} \in]s, T[$ and every $n \in \mathbb{N}$ let $\mathbb{T}_n$ be a partition of $[\tilde{s}, T]$ of the form $\mathbb{T}_n = \{t_{0,n}, \ldots, t_{k_n,n}\}$ for some $k_n \in \mathbb{N}$ and $t_{0,n}, \ldots, t_{k_n,n} \in [\tilde{s}, T]$ with $\tilde{s} = t_{0,n} < \cdots < t_{k_n,n} = T$. We denote its mesh by

$$|\mathbb{T}_n| = \max_{i\in\{0,\ldots,k_n-1\}}(t_{i+1,n} - t_{i,n})$$

and assume that the resulting sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ is refining, which means that $\mathbb{T}_n \subset \mathbb{T}_{n+1}$ for all $n \in \mathbb{N}$, and satisfies $\lim_{n\uparrow\infty} |\mathbb{T}_n| = 0$. Then the sequences $({}_nX)_{n\in\mathbb{N}}$ and $({}_nG(\tau))_{n\in\mathbb{N}}$ of left-continuous $(\mathscr{F}_t)_{t\in[0,T]}$-adapted processes defined via

$${}_nX_t := \sum_{i=0}^{k_n-1} X_{t_{i,n}}\mathbb{1}_{]t_{i,n},t_{i+1,n}]}(t) \quad \text{and} \quad {}_nG_t(\tau) := \sum_{i=0}^{k_n-1} G_{t_{i,n}}(\tau)\mathbb{1}_{]t_{i,n},t_{i+1,n}]}(t)$$

satisfy $\lim_{n\uparrow\infty} {}_nX_t(\omega) = X_t(\omega)$ and $\lim_{n\uparrow\infty} {}_nG_t(\tau)(\omega) = G_t(\tau)(\omega)$ for all $(t, \omega) \in]\tilde{s}, T] \times \Omega$ for which $X(\omega)$ and $G(\tau)(\omega)$ are left-continuous at t. For the decreasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of $[0, T] \cup \{\infty\}$-valued random variables given by

$$\tau_n(\omega) := \sum_{i=0}^{k_n-1} t_{i+1,n}\mathbb{1}_{\{t_{i,n}<\tau\leq t_{i+1,n}\}}(\omega), \quad \text{if } \tau(\omega) < \infty,$$

and $\tau_n(\omega) := \infty$, if $\tau(\omega) = \infty$, we have $\inf_{n\in\mathbb{N}} \tau_n = \tau$ on $\{\tilde{s} < \tau\}$. For given $n \in \mathbb{N}$ we define a left-continuous $(\tilde{\mathscr{F}}_t)_{t\in[0,T]}$-adapted process ${}_n\tilde{X}$ by using the definition of ${}_nX$ when X is replaced by $\tilde{X}$ and compute that

$$E[{}_n\tilde{X}_T^\sigma \mathbb{1}_{\{\tilde{s}<\sigma\leq T\wedge\tau_n\}}|\mathscr{F}_s] = \sum_{i=0}^{k_n-1} E[X_{t_{i,n}} P(t_{i,n} < \sigma \leq t_{i+1,n}, \sigma \leq \tau_n|\mathscr{F}_{t_{i,n}})|\mathscr{F}_s]$$
$$= -E\bigg[\int_{]\tilde{s},T]} {}_nX_t\, {}_nG_t(\tau)\, dG_t(\sigma)\,\bigg|\,\mathscr{F}_s\bigg] \quad \text{a.s.}$$

Indeed, the $(\mathscr{F}_t)_{t\in[0,T]}$-conditional independence of σ and τ gives

$$P(t_{i,n} < \sigma \leq t_{i+1,n}, \sigma \leq \tau_n|\mathscr{F}_{t_{i,n}}) = P(t_{i,n} < \sigma \leq t_{i+1,n}, \tau = \infty|\mathscr{F}_{t_{i,n}})$$
$$+ \sum_{j=i}^{k_n-1} P(t_{i,n} < \sigma \leq t_{i+1,n}, t_{j,n} < \tau \leq t_{j+1,n}|\mathscr{F}_{t_{i,n}})$$
$$= -E[G_{t_{i,n}}(\tau)(G_{t_{i+1,n}}(\sigma) - G_{t_{i,n}}(\sigma))|\mathscr{F}_{t_{i,n}}] \quad \text{a.s.}$$

for each $i \in \{0, \dots, k_n - 1\}$. By construction, $|{}_nX_t| \leq \sup_{\tilde{t}\in]s,T]} |X_{\tilde{t}}|$ for every $n \in \mathbb{N}$ and all $t \in]\tilde{s}, T]$. Therefore, dominated convergence yields that

$$\lim_{n\uparrow\infty} \int_{]\tilde{s},T]} {}_nX_t\, {}_nG_t(\tau)\, dG_t(\sigma) = \int_{]\tilde{s},T]} X_t G_t(\tau)\, dG_t(\sigma).$$

Since $|\int_{]\tilde{s},T]} {}_nX_t\, {}_nG_t(\tau)\, dG_t(\sigma)|$ does not exceed $\sup_{t\in]s,T]} |X_t| G_t(\tau)(V_T(\sigma) - V_s(\sigma))$ for each $n \in \mathbb{N}$, dominated convergence also implies that

$$E[\tilde{X}_T^\sigma \mathbb{1}_{\{\tilde{s}<\sigma\leq T\wedge\tau\}}|\mathscr{F}_s] = -E\bigg[\int_{]\tilde{s},T]} X_t G_t(\tau)\, dG_t(\sigma)\,\bigg|\,\mathscr{F}_s\bigg] \quad \text{a.s.}$$

Finally, for any sequence $(s_n)_{n\in\mathbb{N}}$ in $]s, T[$ that converges to s, we have $\lim_{n\uparrow\infty} \mathbb{1}_{]s_n,T]}(\sigma) = \mathbb{1}_{]s,T]}(\sigma)$ and $\lim_{n\uparrow\infty} \int_{]s,s_n]} X_t G_t(\tau)\, dG_t(\sigma) = 0$. Hence, the claim follows from a final application of the Dominated Convergence Theorem. $\square$

5.2 Proofs for the conditionally independent hitting times

Proof of Lemma 2.7. For any $\omega \in \Omega$ we have $\tau_j(\omega) \leq t$ if and only if $X_t^{(j)}(\omega) \geq \xi_j(\omega)$, since the increasing function $X^{(j)}(\omega)$ is right-continuous. Hence, $\{\tau_j \leq t\}$ coincides with $\{\xi_j \leq X_t^{(j)}\}$ and lies in $\tilde{\mathscr{F}}_t$.

Further, this entails that $\{\tau_1 > s_1, \dots, \tau_j > s_j\} = \{\xi_1 > X_{s_1}^{(1)}, \dots, \xi_j > X_{s_j}^{(j)}\}$. For this reason, the independence of ξ and $\mathscr{F}_T$ and the independence of $\xi_1, \dots, \xi_n$ yield that

$$P(\tau_1 > s_1, \dots \tau_j > s_j|\mathscr{F}_t) = P(\xi_1 > x_1, \dots, \xi_j > x_j)_{|(x_1,\dots,x_n)=(X_{s_1}^{(1)},\dots,X_{s_j}^{(j)})}$$
$$= G_1(X_{s_1}^{(1)}) \cdots G_n(X_{s_j}^{(j)}) \quad \text{a.s.}$$

These considerations imply all the assertions. $\square$

Proof of Lemma 2.8. (i) By (2.5), we have $G_s(\rho) > 0$ a.s. if and only if $G_i(X_s^{(i)}) > 0$ a.s. for any $i \in \{1, \dots, m\}$. Therefore, the definition of the essential supremum implies the claim.

(ii) From the representation $P(\rho > s) = E[\prod_{i=1}^m G_i(X_s^{(i)})]$ we infer that $P(\rho > s) = 1$ if and only if $G_i(X_s^{(i)}) = 1$ a.s. for all $i \in \{1, \dots, m\}$. In addition, $P(\rho > s) = 0$ if only if $G_i(X_s^{(i)}) = 0$ for some $i \in \{1, \dots, m\}$ a.s., which yields the assertions.

(iii) Let $(s_n)_{n\in\mathbb{N}}$ be an increasing sequence in $[0,s[$ that converges to s. Then the σ-continuity of probability measures and dominated convergence lead to

$$P(\rho \geq s) = \lim_{n\uparrow\infty} P(\rho > s_n) = \lim_{n\uparrow\infty} E\Big[\prod_{i=1}^m G_i(X_{s_n}^{(i)})\Big] = E\Big[\prod_{i=1}^m G_i(X_s^{(i)})\Big] = P(\rho > s),$$

since $\lim_{n\uparrow\infty} G_i(X_{s_n}^{(i)}) = G_i(X_s^{(i)})$ a.s. for each $i \in \{1,\dots,m\}$, due to the a.s. left-continuity of $G_i(X^{(i)})$. Hence, $P(\rho = s) = P(\rho \geq s) - P(\rho > s) = 0$. □

Proof of Proposition 2.10. Since $P(\rho > 0) = E[G_0(\rho)] = G_1(\hat{x}_1)\cdots G_m(\hat{x}_m)$ and $\Lambda_0 = \Omega$, we merely need to check the asserted formula for $t > 0$.

For this purpose, we define an $]0,1[$-valued continuously differentiable function φ on $]a_1,b_1[\times\cdots\times]a_m,b_m[$ by $\varphi(x) := \prod_{i=1}^m G_i(x_i)$ and observe that the path

$$]0,t] \to [0,\infty[^m, \quad s \mapsto X_s(\omega)$$

is absolutely continuous and takes all its values in $]a_1,b_1[\times\cdots\times]a_m,b_m[$ for each $\omega \in \Lambda_t$, as $a_i \leq \hat{x}_i < X_s^{(i)}(\omega) \leq X_t^{(i)}(\omega) < b_i$ for every $s \in]0,t]$ and all $i \in \{1,\dots,m\}$. Thus,

$$\varphi(X_s) - \varphi(X_t) = -\sum_{i=1}^m \int_s^t \frac{\partial\varphi}{\partial x_i}(X_{\tilde{s}})\,dX_{\tilde{s}}^{(i)} = -\int_s^t \varphi(X_{\tilde{s}}) \sum_{i=1}^m \lambda_{\tilde{s}}^{(i)} \Big(\frac{G_i'}{G_i}\Big)(X_{\tilde{s}}^{(i)})\,d\tilde{s}$$

on Λ_t, by the Fundamental Theorem of Calculus for Lebesgue-Stieltjes integrals. Now we take expectations and apply Fubini's theorem to the effect that

$$E[\varphi(X_s);\Lambda_t] = P(\rho > t) - \int_s^t E\Big[\varphi(X_{\tilde{s}}) \sum_{i=1}^m \lambda_{\tilde{s}}^{(i)} \Big(\frac{G_i'}{G_i}\Big)(X_{\tilde{s}}^{(i)});\Lambda_t\Big]\,d\tilde{s}. \tag{5.1}$$

Thereby, we used that $E[\varphi(X_t);\Lambda_t] = E[G_t(\rho)\mathbb{1}_{\Lambda_t}] = P(\rho > t)$. From the right-continuity of $G_1,\dots,G_m$ and monotone convergence we infer that

$$G_1(\hat{x}_1)\cdots G_m(\hat{x}_m)P(\Lambda_t) = \lim_{n\uparrow\infty} E[\varphi(X_{s_n});\Lambda_t]$$

for any decreasing zero sequence $(s_n)_{n\in\mathbb{N}}$ in $]0,t]$. Finally, we replace s by s_n in (5.1) for each $n \in \mathbb{N}$ to deduce the claim from monotone convergence. □

5.3 Proofs for the market model with default

Proof of Proposition 3.5. Because (3.12) is equivalent to (C.2), Lemma 2.3 entails for the contractual cash flows and the collateral, funding and hedging costs and benefits that

$$\begin{aligned}\tilde{E}\big[{}_{\mathrm{con}}\mathrm{CF}_s - {}_{\mathrm{col}}\mathrm{C}_s(\mathscr{V}) - {}_{\mathrm{fun}}\mathrm{C}_s(\tilde{\mathscr{V}}) - {}_{\mathrm{hed}}\mathrm{C}_s(\tilde{\mathscr{V}})\big|\mathscr{F}_s\big]&\\ = \tilde{E}\Big[D_{s,T}(r)\Phi(S,V)G_T(\tau) + \int_s^T D_{s,t}(r)\,{}_0\mathrm{B}_t(\mathscr{V})G_t(\tau)\,dt\,\Big|\,\mathscr{F}_s\Big] &\quad \text{a.s.}\end{aligned}$$

Now we consider the cash flows ${}_{\mathrm{def}}\mathrm{CF}(\mathscr{V},\tilde{\mathscr{V}})$ on default, given by (3.10), and note that $\{\tau_i < \tau_j\} \cap \{s < \tau < T\} = \{s < \tau_i < \tau_j \wedge T\}$ for both $i,j \in \{\mathcal{I},\mathcal{C}\}$ with $i \neq j$. By (3.13) and (C.3), we obtain

$$\begin{aligned}\tilde{E}\big[{}_{\mathrm{def}}\mathrm{CF}_s(\mathscr{V},\tilde{\mathscr{V}})\big|\mathscr{F}_s\big] &= -\tilde{E}\Big[\int_s^T D_{s,t}(r)\,{}_{\mathcal{I}}\mathrm{B}_t(\mathscr{V})G_t(\tau_{\mathcal{C}})\,dG_t(\tau_{\mathcal{I}})\,\Big|\,\mathscr{F}_s\Big]\\ &\quad - \tilde{E}\Big[\int_s^T D_{s,t}(r)\,{}_{\mathcal{C}}\mathrm{B}_t(\mathscr{V})G_t(\tau_{\mathcal{I}})\,dG_t(\tau_{\mathcal{C}})\,\Big|\,\mathscr{F}_s\Big] \quad \text{a.s.}\end{aligned}$$

from two applications of Proposition 2.6, since (3.2) ensures $\tau_{\mathcal{I}} \neq \tau_{\mathcal{C}}$ a.s. on $\{\tau < \infty\}$. This shows the assertion. □

Proof of Proposition 3.6. According to our considerations preceding Proposition 3.5, it follows immediately from (C.2) and (C.3) that the integral

$$\int_0^T D_{0,t}(r)\big(|_0\mathrm{B}_t(\mathscr{V})|G_t(\tau)\,dt + |_{\mathcal{I}}\mathrm{B}_t(\mathscr{V})|G_t(\tau_{\mathcal{C}})\,dV_t(\tau_{\mathcal{I}}) + |_{\mathcal{C}}\mathrm{B}_t(\mathscr{V})|G_t(\tau_{\mathcal{I}})\,dV_t(\tau_{\mathcal{C}})\big),$$

which bounds $\int_0^t D_{0,s}(r)\,dA_s(\mathscr{V})$ for any $t \in [0,T]$, is $\tilde{P}$-integrable. This justifies that the process in (3.19) is indeed integrable and we may turn to the second assertion.

For only if we observe that $D_{0,s}(\tau)\mathscr{V}_s G_s(\tau)$ and hence, ${}_{\mathscr{V}}M_s$ is integrable for fixed $s \in [t_0, T]$. Indeed, we get

$$\tilde{E}[D_{0,s}(r)|\mathscr{V}_s|G_s(\tau)] \leq \tilde{E}\bigg[\bigg|D_{0,T}(r)\Phi(S,V)G_T(\tau) + \int_s^T D_{0,t}(r)\,dA_t(\mathscr{V})\bigg|\bigg] < \infty,$$

by using that $D_{0,s}(r)D_{s,t}(r) = D_{0,t}(r)$ for any $t \in [s,T]$. In addition, the $\mathscr{F}_s$-measurability of $\int_0^s D_{0,\tilde{s}}(r)\,dA_{\tilde{s}}(\mathscr{V})$ yields that

$$\begin{aligned} {}_{\mathscr{V}}M_s &= D_{0,s}(r)\mathscr{V}_s G_s(\tau) + \int_0^s D_{0,\tilde{s}}(r)\,dA_{\tilde{s}}(\mathscr{V}) \\ &= \tilde{E}\bigg[D_{0,T}(r)\Phi(S,V)G_T(\tau) + \int_0^T D_{0,t}(r)\,dA_t(\mathscr{V})\,\bigg|\,\mathscr{F}_s\bigg] = \tilde{E}[{}_{\mathscr{V}}M_T|\mathscr{F}_s] \quad \text{a.s.,} \end{aligned}$$

which implies the martingale property of ${}_{\mathscr{V}}M$ relative to $(\mathscr{F}_t)_{t\in[t_0,T]}$. For if the integrability of ${}_{\mathscr{V}}M_s$ entails that of $D_{0,s}(r)\mathscr{V}_s G_s(\tau)$ and we have

$$D_{0,s}(r)\mathscr{V}_s G_s(\tau) = \tilde{E}\bigg[D_{0,T}(r)\mathscr{V}_T G_T(\tau) + \int_s^T D_{0,t}(r)\,dA_t(\mathscr{V})\bigg|\mathscr{F}_s\bigg] \quad \text{a.s.,} \tag{5.2}$$

because ${}_{\mathscr{V}}M_s = \tilde{E}[{}_{\mathscr{V}}M_T|\mathscr{F}_s]$ a.s. Consequently, it holds that

$$\tilde{E}[|\mathscr{V}_s|G_s(\tau)] \leq \tilde{E}\bigg[\bigg|D_{s,T}(r)\Phi(S,V)G_T(\tau) + \int_s^T D_{s,t}(r)\,dA_t(\mathscr{V})\bigg|\bigg] < \infty.$$

This allows us to multiplicate both sides in (5.2) with $D_{0,s}(-r)$ and, since $\mathscr{V}_T G_T(\tau)$ may be replaced by $\Phi(S,V)G_T(\tau)$, we see that $\mathscr{V}$ solves (VE). ☐

Proof of Proposition 3.7. We recall from (3.17) that the continuous process (3.19) is of finite variation, just as the process $[0,T]\times\Omega \to]0,\infty[$, $(t,\omega) \mapsto D_{0,t}(r)(\omega)$. Hence, the first claim follows directly from Itô's formula.

Let us now assume that ${}_{\mathscr{V}}M$ is a continuous $(\mathscr{F}_t)_{t\in[t_0,T]}$-semimartingale and choose $t \in [t_0,T]$. Then from Itô's product rule we infer that

$$\mathscr{V}_t G_t(\tau) - \mathscr{V}_s G_s(\tau) = -\int_s^t \big(dA_{\tilde{s}}(\mathscr{V}) - r_{\tilde{s}}\mathscr{V}_{\tilde{s}} G_{\tilde{s}}(\tau)\,d\tilde{s}\big) + \int_s^t D_{0,\tilde{s}}(-r)\,d{}_{\mathscr{V}}M_{\tilde{s}}$$

for all $s \in [t_0,t]$ a.s., which gives the first identity. In the case that $G(\tau) > 0$ Itô's product rule also shows that $\mathscr{V}$ is a continuous $(\mathscr{F}_t)_{t\in[t_0,T]}$-semimartingale and

$$\mathscr{V}_t - \mathscr{V}_s = \int_s^t \frac{1}{G_{\tilde{s}}(\tau)}\,d\mathscr{V}_{\tilde{s}} G_{\tilde{s}}(\tau) - \int_s^t \frac{\mathscr{V}_{\tilde{s}}}{G_{\tilde{s}}(\tau)}\,dG_{\tilde{s}}(\tau)$$

for every $s \in [t_0,t]$ a.s. Thus, the second identity (3.22) follows from the first (3.21) and the definition of A in (3.17).

Finally, suppose there is a continuous $(\mathscr{F}_t)_{t\in[t_0,T]}$-semimartingale M such that (3.22) holds when ${}_{\mathscr{V}}M$ is replaced by M. Then from Itô's formula we obtain that

$${}_{\mathscr{V}}M_t - {}_{\mathscr{V}}M_s = -\int_s^t D_{0,\tilde{s}}(r)G_{\tilde{s}}(\tau)\mathscr{V}_{\tilde{s}}\Big(r_{\tilde{s}}\,d\tilde{s} - \frac{1}{G_{\tilde{s}}(\tau_{\mathcal{I}})}\,dG_{\tilde{s}}(\tau_{\mathcal{I}}) - \frac{1}{G_{\tilde{s}}(\tau_{\mathcal{I}})}\,dG_{\tilde{s}}(\tau_{\mathcal{C}})\Big)$$

$$+\int_s^t D_{0,\tilde{s}}(r)G_{\tilde{s}}(\tau)\Big(d\mathscr{V}_{\tilde{s}} + \frac{1}{G_{\tilde{s}}(\tau)}dA_{\tilde{s}}(\mathscr{V})\Big) = M_t - M_s$$

for each $s \in [t_0, t]$ a.s. This shows that if $M_{t_1} = {}_{\mathscr{V}}M_{t_1}$ a.s., then M and ${}_{\mathscr{V}}M$ must be indistinguishable, as claimed. □

Proof of Corollary 3.8. For only if $D_{0,t_0}(r)\mathscr{V}_{t_0}G_{t_0}(\tau)$ is integrable, since the process (3.19) satisfies this property and ${}_{\mathscr{V}}M$ is a continuous $(\mathscr{F}_t)_{t\in[t_0,T]}$-martingale, by Proposition 3.6. Moreover, Proposition 3.7 states that (3.22) is valid for all $s \in [t_0, T]$ a.s.

For if $\mathscr{V}$ is an $(\mathscr{F}_t)_{t\in[t_0,T]}$-semimartingale, since (3.22) holds for any $s \in [t_0, T]$ a.s. when ${}_{\mathscr{V}}M$ is replaced by the $(\mathscr{F}_t)_{t\in[t_0,T]}$-martingale $M - M_{t_0} + {}_{\mathscr{V}}M_{t_0}$. Hence, the uniqueness assertion of Proposition 3.7 yields $M_t - M_{t_0} = {}_{\mathscr{V}}M_t - {}_{\mathscr{V}}M_{t_0}$ for each $t \in [t_0, T]$ a.s. and Proposition 3.6 shows that $\mathscr{V}$ solves (VE). □

6 Proofs for the volatility model and the valuation PDE

6.1 Proofs for the price process and its quasi variance

Proof of Lemma 4.1. Regarding uniqueness, suppose that S and $\tilde{S}$ are two solutions to the first SDE in (4.1) with $S_{t_0} = \tilde{S}_{t_0}$ a.s. For fixed $n \in \mathbb{N}$ there is $k_\theta \in \mathscr{L}^2(\mathbb{R}_+)$ such that $|\theta(\cdot, v)| \le k_\theta$ for any $v \in [1/n, n]$ a.e. Thus, Itô's formula yields that

$$E\big[|S_t^{\tau_n} - \tilde{S}_t^{\tau_n}|^2\big] \le \int_{t_0}^t (2b^+ + k_\theta^2)(s)E\big[|S_s^{\tau_n} - \tilde{S}_s^{\tau_n}|^2\big]\,ds \quad \text{for all } t \in [t_0, T]$$

and the stopping time $\tau_n := \inf\{t \in [t_0, T] \,|\, V_t \notin]1/n, n[\text{ or } |S_t| \vee |\tilde{S}_t| \ge n\}$. By Gronwall's inequality and $\sup_{n\in\mathbb{N}} \tau_n = \infty$, the continuous processes S and $\tilde{S}$ are indistinguishable.

Regarding existence and the claimed representation, note that $\int_{t_0}^T \theta(s, V_s)^2\,ds < \infty$, as $V([t_0, T] \times \{\omega\})$ is compact in $]0, \infty[$ for any $\omega \in \Omega$. For this reason, the stochastic and Lebesgue integrals in (4.2) are well-defined. Hence, if S is an adapted continuous process for which (4.2) holds, then Itô's formula shows that it solves the first SDE in (4.1).

The second claim is a direct consequence of Novikov's condition, which entails that the continuous local martingale $\exp(\int_{t_0}^{\cdot} \theta(s, V_s)\,d\hat{W}_s - \frac{1}{2}\int_{t_0}^{\cdot} \theta(s, V_s)^2\,ds)$ is a martingale and hence,

$$E[S_t]e^{-\int_{t_0}^t b(s)\,ds} = E\bigg[\chi E\bigg[\exp\bigg(\int_{t_0}^t \theta(s, V_s)\,d\hat{W}_s - \frac{1}{2}\int_{t_0}^t \theta(s, V_s)^2\,ds\bigg)\bigg|\mathscr{F}_{t_0}\bigg]\bigg] = E[\chi]$$

for all $t \in [t_0, T]$. Thereby, we used the fact that S_t is integrable, which follows from the same reasoning if χ is split into its positive and negative part. □

Proof of Lemma 4.2. By using the sublinear growth and Hölder condition for θ, we notice that $|\theta(\cdot, v)^2 - \theta(\cdot, \tilde{v})^2| \le (2k_\theta + \lambda_\theta(v^{1/2} + \tilde{v}^{1/2}))\lambda_\theta|v - \tilde{v}|^{1/2}$ for any $v, \tilde{v} > 0$ a.e. Hence, the Cauchy-Schwarz inequality implies that

$$\frac{1}{2}E\bigg[\int_{t_0}^{t\wedge\sigma} |\theta(s, V_s)^2 - \theta(s, \tilde{V}_s)^2|\,ds\bigg] \le c_{2,1}(t) \sup_{s\in[t_0,t]} (1 + E[V_s^\sigma] + E[\tilde{V}_s^\sigma])^{\frac{1}{2}} E\big[|V_s^\sigma - \tilde{V}_s^\sigma|\big]^{\frac{1}{2}}$$

with $c_{2,1} : [t_0, T] \to \mathbb{R}_+$ given by $c_{2,1}(t) := \int_{t_0}^t (k_\theta(s) + \lambda_\theta(s))\lambda_\theta(s)\,ds$. Moreover, from the Burkholder-Davis-Gundy inequality we infer that

$$E\bigg[\sup_{\tilde{s}\in[t_0,t]} \bigg|\int_{t_0}^{\tilde{s}\wedge\sigma} \theta(s, V_s) - \theta(s, \tilde{V}_s)\,d\hat{W}_s\bigg|\bigg] \le c_{2,2}(t) \sup_{s\in[t_0,t]} E\big[|V_s^\sigma - \tilde{V}_s^\sigma|\big]^{\frac{1}{2}},$$

where $c_{2,2} : [t_0, T] \to \mathbb{R}_+$ is defined via $c_{2,2}(t) := 2(\int_{t_0}^t \lambda_\theta(s)^2\,ds)^{1/2}$. Because we have $c_2(t_0, \cdot) = c_{2,1} + c_{2,2}$, the desired estimate follows. □

The proof of Proposition 4.4, which extends several ideas from the proof of Theorem 2.2 in [29], relies on the construction of the following function.

Lemma 6.1. *Under* (V.7), *there is* $\varphi \in C^1(]0,\infty[)$ *satisfying the following two conditions:*

(i) $\lim_{x\downarrow 0}\varphi(x) = -\infty$ *and* $\varphi' > 0$. *Further,* φ *is twice continuously differentiable on* $]0,\varepsilon]$ *and* $[\varepsilon,\infty[$ *such that* $\varphi''(x) > 0$ *for any* $x > \varepsilon$.

(ii) $((\eta^2/2)\varphi'' + \zeta\varphi')(\cdot,x)$ *is bounded from below by* $-c_0\varphi_0(x)$, *whenever* $x < \varepsilon$, *and by* $-c_\zeta\varphi_\zeta(x)\varphi'(x)$, *if* $x > \varepsilon$, *for all* $x > 0$ *with* $x \neq \varepsilon$ *a.e.*

Proof. We define $\hat{\varphi} \in C^1([\varepsilon,\infty[)$ by $\hat{\varphi}(x) := (1/\varepsilon)\exp(\int_\varepsilon^x \varphi_\zeta(y)^{-1}\,dy)$. Then it follows readily that the function $\varphi :]0,\infty[\to \mathbb{R}$ given by $\varphi(x) := \log(x)$ for $x < \varepsilon$ and

$$\varphi(x) := \log(\varepsilon) + \int_\varepsilon^x \hat{\varphi}(y)\,dy \quad \text{for } x \geq \varepsilon$$

is a feasible choice, which, however, is not of class C^2, since $\varphi''_+(\varepsilon) = 1/(\varphi_\zeta(\varepsilon)\varepsilon)$. □

Proof of Proposition 4.4. The assertion follows if we can verify that the stopping time $\sigma := \inf\{t \in [t_0,T]\,|\,V_t \leq 0\}$ satisfies $\sigma = \infty$ a.s. To this end, we choose $n_0 \in \mathbb{N}$ with $v_0/n_0 < \varepsilon < n_0 v_0$ and introduce two stopping times by

$$\sigma_m := \inf\{t \in [t_0,T]\,|\,V_t \leq v_0/m\} \quad \text{and} \quad \overline{\sigma}_n := \inf\{t \in [t_0,T]\,|\,V_t \geq n v_0\}$$

for $m,n \in \mathbb{N}$ with $m \wedge n \geq n_0$. Further, let $\varphi \in C^1(]0,\infty[)$ satisfy the conditions of Lemma 6.1 and set $\varphi(x) := 0$ for any $x \in]-\infty,0]$ and $B_\varepsilon(\omega) := \{t \in [t_0,T]\,|\,V_t(\omega) \neq \varepsilon\}$ for all $\omega \in \Omega$. Then the generalised Itô formula in [33] gives

$$\begin{aligned}\varphi(V_T^{\sigma_{m,n}}) - \varphi(v_0) &= \int_{t_0}^{T\wedge\sigma_{m,n}} \varphi'(V_s)\,dV_s + \frac{1}{2}\int_{B_\varepsilon} \varphi''(V_s)\mathbb{1}_{[t_0,\sigma_{m,n}]}(s)\,d\langle V\rangle_s \\ &\geq -k_\varepsilon - \varphi_\zeta(nv_0)\varphi'(nv_0)\int_{t_0}^T c_\zeta(s)\,ds + \int_{t_0}^{T\wedge\sigma_{m,n}} \varphi'(V_s)\eta(s,V_s)\,d\tilde{W}_s \quad \text{a.s.,}\end{aligned}$$

where $\sigma_{m,n} := \sigma_m \wedge \overline{\sigma}_n$ and $k_\varepsilon := \varphi_0(\varepsilon)\int_{t_0}^T c_0(s)\,ds + \varphi_\zeta(\varepsilon)\varphi'(\varepsilon)\int_{t_0}^T c_\zeta(s)\,ds$. For any $l \in \mathbb{N}$ we define a stopping time by $\hat{\sigma}_l := \inf\{t \in [t_0,T]\,|\,\int_{t_0}^t \varphi'(V_s)^2\eta(s,V_s)^2\,ds \geq l\}$. Then

$$E[\varphi(V_T^{\sigma_{m,n}})] = \lim_{l\uparrow\infty} E[\varphi(V_T^{\hat{\sigma}_l\wedge\sigma_{m,n}})] \geq \varphi(v_0) - k_\varepsilon - \varphi_\zeta(nv_0)\varphi'(nv_0)\int_{t_0}^T c_\zeta(s)\,ds$$

follows from dominated convergence. At the same time, as $V_{\sigma_m} = v_0/m$ a.s. on $\{\sigma_m < \infty\}$ and $\lim_{x\downarrow 0}\varphi(x) = -\infty$, we obtain that

$$E[\varphi(V_T^{\sigma_{m,n}})] \leq \varphi(v_0/m)P(\sigma_m \leq \overline{\sigma}_n \wedge T) + \max_{x\in]0,nv_0]} \varphi^+(x).$$

These two estimates imply that $\lim_{m\uparrow\infty} P(\sigma_m \leq \overline{\sigma}_n \wedge T) = P(\sigma \leq \overline{\sigma}_n \wedge T) = 0$. Hence, $\sigma > \overline{\sigma}_n \wedge T$ a.s. and the desired result follows from the fact that $\sup_{n\in\mathbb{N}} \overline{\sigma}_n = \infty$. □

Proof of Proposition 4.6. (i) The drift and diffusion coefficients of the SDE (4.11) are independent of the first spatial coordinate. For this reason, the claim follows directly from Corollary 3.9, Remark 3.10 and Proposition 3.13 in [26], which yield pathwise uniqueness for the second SDE in (4.1).

(ii) From Theorem 3.27 in [26] we know that there is a unique strong solution V^{t_0,v_0} to the second SDE in (4.1) such that $V_{t_0}^{t_0,v_0} = v_0$ a.s. According to Proposition 4.4, the positivity of $V^{t_0,v_0}(\omega)$ for P-a.e. $\omega \in \Omega$ holds and, by taking a continuous modification if necessary, we may assume that $V^{t_0,v_0}(\omega) > 0$ for all $\omega \in \Omega$.

Furthermore, we may choose a continuous process X^{t_0,x_0,v_0} that is adapted to the augmented natural filtration of the two-dimensional $(\mathscr{F}_t)_{t\in[0,T]}$-Brownian motion $(W,\tilde{W})$ and satisfies (4.3) for $V = V^{t_0,v_0}$. This argumentation justifies the assertion.

(iii) For any sequence $(s_n,x_n,v_n)_{n\in\mathbb{N}}$ and each point (s,x,v) in $[0,t]\times\mathbb{R}\times]0,\infty[$ such that $s\leq s_n$ for all $n\in\mathbb{N}$ and $\lim_{n\uparrow\infty}(s_n,x_n,v_n)=(s,x,v)$, Theorem 21.4 and Lemma 21.9 in [1] yield

$$\lim_{n\uparrow\infty} E\big[\varphi(s_n,X_t^{s_n,x_n,v_n},V_t^{s_n,v_n})\big] = E\big[\varphi(s,X_t^{s,x,v},V_t^{s,v})\big]$$

if $(X_t^{s_n,x_n,v_n},V_t^{s_n,v_n})_{n\in\mathbb{N}}$ converges in probability to $(X_t^{s,x,v},V_t^{s,v})$. By using Lemma 4.2, this can be inferred from Theorem 4.6 in [26], which gives a first moment estimate for random Itô processes and implies the two estimates (4.6) and (4.7).

(iv) According to Lemma 3.5 in [24], for instance, the Markov property of the triple $((\hat{X},\hat{V}),(\hat{\mathscr{F}}_t)_{t\in[0,T]},\mathbb{P})$ holds if we can show that

$$E_{s,x,v}\big[\varphi(\hat{X}_t,\hat{V}_t)|\hat{\mathscr{F}}_{\tilde{s}}\big] = E_{\tilde{s},\hat{X}_{\tilde{s}},\hat{V}_{\tilde{s}}}\big[\varphi(\hat{X}_t,\hat{V}_t)\big] \quad P_{s,x,v}\text{-a.s.}$$

for any $s,\tilde{s},t\in[0,T]$ with $s\leq\tilde{s}\leq t$, each $(x,v)\in\mathbb{R}\times]0,\infty[$ and every Lipschitz continuous function $\varphi:\mathbb{R}\times]0,\infty[\to[0,1]$. This follows from the same reasoning as in the proof of Theorem 5.1.5 in [34].

Finally, as Lemma 3.14 in [24] shows that any Markov process with right-continuous paths that is right-hand Feller is also strongly Markov, the proof is complete. ☐

6.2 Proofs for the valuation function as mild and classical solution

Proof of Theorem 4.10. (i) Let $(s,x,v)\in[0,T]\times\mathbb{R}\times]0,\infty[$ and (X,V) be a solution to (4.11) on $[s,T]$ with $(X^s,V^s)=(x,v)$ a.s., in which case it also solves (4.19) under $\tilde{P}$. In particular, $\tilde{P}_{s,x,v}$ must be its law under $\tilde{P}$. Hence, (4.14) and (4.15) hold and our discussion preceding (4.16) shows that (M.1)-(M.4) are valid. For this reason, we may consider solutions to (VE).

Since α, β, $\hat{H}$ and u are continuous, so are the collateral process $\alpha\mathscr{V}$, the pre-default funding process $(1-\alpha)\mathscr{V}$, the pre-default hedging process $\hat{H}(\cdot,\exp(X),V,\mathscr{V})$ and the close-out value $\beta\mathscr{V}$, which justifies that (C.1) holds.

By Remark 4.8, the conditions (C.2) and (C.3) follow from (4.24) and the boundedness of u. For this reason, Propositions 3.6 and 3.7 entail that $\mathscr{V}$ is an $(\mathscr{F}_t)_{t\in[s,T]}$-semimartingale and the process $M:[s,T]\times\Omega\to\mathbb{R}$ given by

$$M_t := D_{0,t}(r)\mathscr{V}_t G_t(\tau) + \int_0^t D_{0,\hat{s}}(r)G_{\hat{s}}(\tau)\big(\hat{B}(\hat{s},\exp(X_{\hat{s}}),V_{\hat{s}},\mathscr{V}_{\hat{s}}) + (\hat{r}-g_{\mathcal{I}}-g_{\mathcal{C}})(\hat{s})\mathscr{V}_{\hat{s}}\big)\,d\hat{s}$$

is a continuous $(\mathscr{F}_t)_{t\in[s,T]}$-martingale under $\tilde{P}$ such that

$$\mathscr{V}_t = \phi(\exp(X_T),V_T) + \int_t^T \hat{B}(\hat{t},\exp(X_{\hat{t}}),V_{\hat{t}},\mathscr{V}_{\hat{t}})\,d\hat{t} - \int_t^T \frac{D_{0,\hat{t}}(-r)}{G_{\hat{t}}(\tau)}\,dM_{\hat{t}} \quad \text{a.s.} \tag{6.1}$$

for all $t\in[s,T]$ a.s. Thereby we used the definitions of the two random functionals A, B and the function $\hat{B}$ in (3.17), (3.24) and (4.16) to find the relation

$$\dot{A}_t(Y) = G_t(\tau)\big(\hat{B}(t,\exp(X_t),V_t,Y) + (\hat{r}-g_{\mathcal{I}}-g_{\mathcal{C}})(t)Y\big) \tag{6.2}$$

for a.e. $t\in[0,T]$ a.s. for any continuous $Y\in\mathscr{S}$. Because $\phi(\exp(X_T),V_T)$ is bounded and $\int_s^T|\hat{B}(t,\exp(X_t),V_t,\mathscr{V}_t)|\,dt$ is integrable, we obtain that

$$\tilde{E}\bigg[\sup_{t\in[s,T]}\bigg|\int_s^t \frac{D_{0,\hat{s}}(-r)}{G_{\hat{s}}(\tau)}\,dM_{\hat{s}}\bigg|\bigg] < \infty.$$

In particular, the $(\mathscr{F}_t)_{t\in[s,T]}$-local martingale $\int_s^\cdot D_{0,\hat{s}}(-r)G_{\hat{s}}(\tau)^{-1}\,dM_{\hat{s}}$ is of class (DL) and therefore, it must be a martingale. Now we may take expectations in (6.1) to see that u satisfies (4.22).

(ii) We already know that (X,V) solves (4.19) on $[s,T]$ under $\tilde{P}$. Thus, the condition on the filtration entails that the process $N:[s,T]\times\Omega\to\mathbb{R}$ defined via

$$N_t := \mathscr{V}_t + \int_s^t \hat{B}(\hat{s},\exp(X_{\hat{s}}),V_{\hat{s}},\mathscr{V}_{\hat{s}})\,d\hat{s}$$

is a continuous $(\mathscr{F}_t)_{t\in[s,T]}$-martingale under $\tilde{P}$. Indeed, its integrability follows directly from the definition of a mild solution. The martingale property $\tilde{E}[N_t|\mathscr{F}_{\tilde{s}}] = N_{\tilde{s}}$ a.s. for any $\tilde{s},t\in[s,T]$ with $\tilde{s}\leq t$ is implied by

$$\tilde{E}\bigg[\mathscr{V}_t + \int_{\tilde{s}}^t \hat{B}(\hat{s},\exp(X_{\hat{s}}),V_{\hat{s}},\mathscr{V}_{\hat{s}})\,d\hat{s}\,\bigg|\,\mathscr{F}_{\tilde{s}}\bigg] = \mathscr{V}_{\tilde{s}} \quad \text{a.s.}$$

This identity is a consequence of Proposition 4.6, by extending the Markov property of the diffusion with measure theoretical methods. See Proposition 3.7 in [24][Chapter 3], for example. In particular, $\mathscr{V}$ is an $(\mathscr{F}_t)_{t\in[s,T]}$-semimartingale.

Hence, for any continuous $(\mathscr{F}_t)_{t\in[s,T]}$-local martingale M that is indistinguishable from the stochastic integral $\int_s^\cdot D_{0,\hat{s}}(r)G_{\hat{s}}(\tau)\,dN_{\hat{s}}$ the representation (6.1) holds. Now we obtain that

$$M_t + D_{0,s}(r)u(s,x,v)G_s(\tau) = D_{0,t}(r)\mathscr{V}_tG_t(\tau) + \int_s^t D_{0,\hat{s}}(r)\,dA_{\hat{s}}(\mathscr{V})$$

for each $t\in[s,T]$ a.s. by the uniqueness assertion of Proposition 3.7. The boundedness of u and the integrability of the random variable $\int_s^T D_{0,\hat{s}}(r)|\dot{A}_{\hat{s}}(\mathscr{V})|\,d\hat{s}$ imply that

$$\tilde{E}\bigg[\sup_{t\in[s,T]}\bigg|D_{0,t}(r)\mathscr{V}_tG(\tau) + \int_0^t D_{0,\hat{s}}(r)\,dA_{\hat{s}}(\mathscr{V})\bigg|\bigg] < \infty.$$

Thus, $D_{0,\cdot}(r)\mathscr{V}G(\tau)+\int_0^\cdot D_{0,\hat{s}}(r)\,dA_{\hat{s}}(\mathscr{V})$ is a continuous $(\mathscr{F}_t)_{t\in[s,T]}$-local martingale that is of class (DL). Therefore, it is a martingale and we may invoke Proposition 3.6 to complete the proof. □

Proof of Proposition 4.12. By (P.1), (P.2) and (P.4), all the rates $\hat{r}$, $\hat{c}_+$, $\hat{c}_-$, $\hat{f}_+$, $\hat{f}_-$, $\hat{h}_+$, $\hat{h}_-$ and the functions $g_{\mathcal{I}}$, $g_{\mathcal{C}}$, α, β are integrable. Thus, (P.5) ensures that the affine growth condition (4.25) and the Lipschitz condition (4.26) on compacts sets holds for $\hat{B}$.

Moreover, we note that $\hat{B}(t,\cdot,\cdot,\cdot)$ is continuous for a.e. $t\in[0,T]$, since $\hat{\pi}$ and $\hat{H}$ are continuous, according to (P.3) and (P.4). Hence, as we know that Proposition 4.6 is applicable, all requirements of Theorem 2.15 in [25] are met and the assertions follow. □

Proof of Corollary 4.14. By Proposition 4.6 and our discussion preceding the corollary, the assumptions (2.2), (2.3) and (2.9)-(2.13) in [2] hold. Further, from (P.5) and (P.6) we infer that $\hat{B}$ is of affine growth and locally Lipschitz continuous in $y\in\mathbb{R}$, uniformly in $(t,s,v)\in[0,T]\times]0,\infty[^2$, and it is also locally Hölder continuous. This shows that the hypotheses (2.19), (2.20) and (2.21) in [2] are satisfied and the assertion follows from Theorem 2.4 in this reference. □

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The normal approximation of compound Hawkes functionals

Mahmoud Khabou

December 2022

We derive quantitative bounds in the Wasserstein distance for the approximation of stochastic integrals of deterministic and non-negative integrands with respect to Hawkes processes by a normally distributed random variable. Our results are specifically applied to compound Hawkes processes, and improve on the current literature where estimates may not converge to zero in large time, or have been obtained only for specific kernels such as the exponential or Erlang functions.

Dynamic programming principle and computable prices in financial market models with transaction costs

Emmanuel Lépinette

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How to compute (super) hedging costs in rather general financial market models with transaction costs in discrete-time ? Despite the huge literature on this topic, most of results are characterizations of the super-hedging prices while it remains difficult to deduce numerical procedure to estimate them. We establish here a dynamic programming principle and we prove that it is possible to implement it under some conditions on the conditional supports of the price and volume processes for a large class of market models including convex costs such as order books but also non convex costs, e.g. fixed cost models.

Joint work with Vu Duc Think.

Joint SPX--VIX calibration with Gaussian polynomial volatility models: deep pricing with quantization hints

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We consider the joint SPX-VIX calibration within a general class of *Gaussian polynomial volatility models* in which the volatility of the SPX is assumed to be a polynomial function of a Gaussian Volterra process defined as a stochastic convolution between a kernel and a Brownian motion. By performing joint calibration to daily SPX-VIX implied volatility surface data between 2012 and 2022, we compare the empirical performance of different kernels and their associated Markovian and non-Markovian models, such as rough and non-rough path-dependent volatility models. In order to ensure an efficient calibration and a fair comparison between the models, we develop a generic unified method in our class of models for fast and accurate pricing of SPX and VIX derivatives based on functional quantization and Neural Networks. For the first time, we identify a *conventional one-factor Markovian continuous stochastic volatility model* that is able to achieve remarkable fits of the implied volatility surfaces of the SPX and VIX together with the term structure of VIX futures. What is even more remarkable is that our *conventional one-factor Markovian continuous stochastic volatility model* outperforms, in all market conditions, its rough and non-rough path-dependent counterparts with the same number of parameters.

Joint work with Eduardo Abi Jaber and Camille Illand

Fractional integral equations with weighted Takagi-Landsberg functions

Vitalii Makogin

January 6, 2023

In the talk, we find fractional Riemann-Liouville derivatives for the Takagi-Landsberg functions. Moreover, we introduce their generalizations called weighted Takagi-Landsberg functions. Namely, for constants $c_{m,k} \in [-L, L]$, $k, m \in \mathbb{N}_0$, we define a *weighted Takagi-Landsberg function* as $y_{c,H} : [0,1] \to \mathbb{R}$ via

$$y_{c,H}(t) = \sum_{m=0}^{\infty} 2^{m\left(\frac{1}{2}-H\right)} \sum_{k=0}^{2^m-1} c_{m,k} e_{m,k}(t), t \in [0,1],$$

where $H > 0$, $\{e_{m,k}, m \in \mathbb{N}_0, k = 0, \ldots, 2^m - 1\}$ are the Faber-Schauder functions on [0,1]. The class of the weighted Takagi-Landsberg functions of order $H > 0$ on $[0,1]$ coincides with the H-Hölder continuous functions on $[0,1]$. Based on computed fractional integrals and derivatives of the Haar and Schauder functions, we get a new series representation of the fractional derivatives of a Hölder continuous function. This result allows to get the new formula of a Riemann-Stieltjes integral. The application of such series representation is the new method of numerical solution of the Volterra and linear integral equations driven by a Hölder continuous function.

Misfortunes Never Come Singly: Managing the Risk of Chain Disasters

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December 20, 2022

Abstract

The paper studies optimal disaster prevention and growth policies in an environment where the arrivals of primary disasters trigger subsequent shocks through contagion effects. To model the interrelated disasters, we use the Hawkes process, which is a novelty in general equilibrium economics. We derive analytical solutions for the optimal growth path and an optimal mitigation policy. We find that the existence of interrelationships between different shocks makes optimal disaster spending stochastic, which is in contrast to the previous literature that advocates a constant share of income for disaster mitigation. An efficient abatement policy depends positively on the arrival rate of the primary shock and jumps upwards when an initial disaster occurs. Such behavior is consistent with the evidence on economy-wide aid during the recent COVID-19 pandemic. We extend the analysis by including Brownian uncertainty and random catastrophe magnitude in the Hawkes process, which shows the versatility of our approach.

Convex stochastic optimization

Teemu Pennanen

King's College London

We study dynamic programming, duality and optimality conditions in general convex stochastic optimization problems introduced by Rockafellar and Wets in the 70s. We give a general formulation of the dynamic programming recursion and derive an explicit dual problem in terms of two dual variables, one of which is the shadow price of information while the other one gives the marginal cost of a perturbation much like in classical Lagrangian duality. Existence of primal solutions and the absence of duality gap are obtained without compactness or boundedness assumptions. In the context of financial mathematics, the relaxed assumptions are satisfied under the well-known no-arbitrage condition and the reasonable asymptotic elasticity condition of the utility function. We extend classical portfolio optimization duality theory to problems of optimal semi-static hedging. Besides financial mathematics, we obtain several new results in stochastic programming and stochastic optimal control.

Mean Field Optimization Problem Regularized by Fisher Information

Abstract: Recently there is a rising interest in the research of mean-field optimization, in particular because of its role in analyzing the training of neural networks. In this talk, by adding the Fisher Information (in other word, the Schrodinger kinetic energy) as the regularizer, we relate the mean-field optimization problem with a so-called mean field Schrodinger (MFS) dynamics. We develop a free energy method to show that the marginal distributions of the MFS dynamics converge exponentially quickly towards the unique minimizer of the regularized optimization problem. We shall see that the MFS is a gradient flow on the probability measure space with respect to the relative entropy. Finally we propose a Monte Carlo method to sample the marginal distributions of the MFS dynamics. This is an ongoing joint work with Julien Claisse, Giovanni Conforti and Songbo Wang.

ROBUST ASYMPTOTIC INSURANCE-FINANCE ARBITRAGE

THORSTEN SCHMIDT

Abstract. In most cases, insurance contracts are linked to the financial markets, such as through interest rates or equity-linked insurance products. To motivate an evaluation rule in these hybrid markets, (Artzner, Eisele, Schmidt, 2022) introduced the notion of insurance-finance arbitrage. We extend their setting by incorporating model uncertainty. To this end, we allow statistical uncertainty in the underlying dynamics to be represented by a set of priors $\mathscr{P}$. Within this framework we introduce the notion of *robust asymptotic insurance-finance arbitrage* and characterize the absence of such strategies in terms of the concept of $Q\mathscr{P}$-evaluations. This is a nonlinear two-step evaluation which guarantees *no robust asymptotic insurance-finance arbitrage*. Moreover, the $Q\mathscr{P}$-evaluation dominates all two-step evaluations as long as we agree on the set of priors $\mathscr{P}$ which shows that those two-step evaluations do not allow for robust asymptotic insurance-finance arbitrages.

This is joint work with Katharina Oberpriller and Moritz Ritter
Published at: `arXiv:2207.13350`

Approximation of PDEs on Wasserstein space and application to mean field control

Mehdi Talbi

ETH Zürich

We present a finite-dimensional approximation for a class of partial differential equations on the space of probability measures. These equations are satisfied in the sense of viscosity solutions. Our main result states the convergence of the viscosity solutions of the finite-dimensional PDE to the viscosity solutions of the PDE on Wasserstein space, provided that uniqueness holds for the latter, and heavily relies on an adaptation of the Barles & Souganidis monotone scheme to our context. We then apply this result to the Hamilton-Jacobi-Bellman and Bellman-Isaacs equations arising in stochastic control and differential games, for which we show the convergence of the value function of the finite population problem to the value function of the mean field problem under rather weak regularity requirements.

Ergodic robust maximization of asymptotic growth under stochastic volatility

Josef Teichmann (ETH)

We consider an asymptotic robust growth problem under model uncertainty and in the presence of (non-Markovian) stochastic covariance. We fix two inputs representing the instantaneous covariance for the asset process X, which depends on an additional stochastic factor process Y, as well as the invariant density of X together with Y. The stochastic factor process Y has continuous trajectories but is not even required to be a semimartingale. Our setup allows for drift uncertainty in X and model uncertainty for the local dynamics of Y. This work builds upon a recent paper of Kardaras & Robertson, where the authors consider an analogous problem, however, without the additional stochastic factor process. Under suitable, quite weak assumptions, we are able to characterize the robust optimal trading strategy and the robust optimal growth rate. The optimal strategy is shown to be functionally generated and, remarkably, does not depend on the factor process Y. Our result provides a comprehensive answer to a question proposed by Fernholz in 2002. Mathematically, we use a combination of partial differential equation (PDE), calculus of variations and generalized Dirichlet form techniques. This is a joint work with David Itkin, Benedikt Koch, and Martin Larsson.

Noncommutative martingale inequalities

Quanhua Xu

Université de Franche-Comté

We will present the noncommutative analogues of the classical Burkholder-Gundy square function inequalities, as well as a bref introduction to Ito-Clifford stochastic integral.

VOLTERRA SANDWICHED VOLATILITY MODEL: MARKOVIAN APPROXIMATION AND HEDGING

G. DI NUNNO[1] AND A. YURCHENKO-TYTARENKO[2]

We propose a new market model with a stochastic volatility driven by a general Hölder continuous Gaussian Volterra process, i.e. the resulting price is not a Markov process. On the one hand, it is consistent with empirically observed phenomenon of market memory, but, on the other hand, brings a vast amount of issues of a technical nature, especially in optimization problems. In the talk, we describe a way to obtain a Markovian approximation to the model as well as exploit it for the numerical computation of the optimal hedge. Two numerical methods are considered: Nested Monte Carlo and Least Squares Monte Carlo. The results are illustrated by simulations.

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The present research is carried out within the frame and support of the ToppForsk project nr. 274410 of the Research Council of Norway with title STORM: Stochastics for Time-Space Risk Models.